2. **Rational solutions of** $x^2 + y^2 = z^2$. Diophantus (*Arithmetica*, II, 8) took in effect $y = x(m/n) - z$, where m/n is a fraction in its lowest terms [the value of $(y + z)/x$]. Thus

$$x = \frac{2mnz}{m^2 + n^2}, \qquad y = \frac{xm}{n} - z = \frac{(m^2 - n^2)z}{m^2 + n^2}.$$

Hence *the sides of any rational right triangle are proportional to*

$$2mn, \qquad m^2 - n^2, \qquad m^2 + n^2, \tag{1}$$

where m and n are relatively prime positive integers.[1] Diophantus spoke of the right triangle with the sides (1) as that *formed from m, n.*

3. **Integral solutions of** $x^2 + y^2 = z^2$. We shall prove that *all positive integral solutions of $x^2 + y^2 = z^2$ are given without duplication by*

$$x = 2mnl, \qquad y = (m^2 - n^2)l, \qquad z = (m^2 + n^2)l, \qquad m > n > 0, \tag{2}$$

where m and n are relatively prime integers and not both odd, while l is a positive integer.[2]

When m and n are both odd, the numbers (1) rearranged are the doubles of the numbers $\frac{1}{2}(m^2 - n^2)$, mn, $\frac{1}{2}(m^2 + n^2)$, which are the sides of the right triangle formed from the integers $\frac{1}{2}(m + n)$, $\frac{1}{2}(m - n)$. Hence we may restrict attention to the case in which m and n are relatively prime and not both odd, so that one is even and the other odd. Thus the last two numbers (1) are odd; any common divisor of those two would divide their sum $2m^2$ and their difference $2n^2$, and hence divide m^2 and n^2, which are relatively prime. Hence the last two numbers (1) have no common divisor > 1. Thus if their products by the same irreducible fraction a/b are both integers, they must be divisible by b, whence $b = 1$. Hence integers are the only rational numbers whose products by all the numbers (1) give integers, when m and n are relatively prime and not both odd. By Diophantus's result in § 2, all integral solutions (like all rational solutions) are products of the numbers (1) by rational numbers. To complete the proof of our present theorem, it remains only to show that the integral solutions (2) contain no duplicates.

Suppose that the numbers (2) coincide with

$$2MNL, \qquad (M^2 - N^2)L, \qquad (M^2 + N^2)L,$$

[1] To give another interesting proof, we note that, if θ is an angle of any rational right triangle, $\sin\theta$ and $\cos\theta$ are rational and hence $t = \tan\frac{1}{2}\theta = \sin\theta/(1 + \cos\theta)$ is rational. Conversely, if t is rational, also

$$\sin\theta = \frac{2t}{1 + t^2}, \qquad \cos\theta = \frac{1 - t^2}{1 + t^2}$$

are rational. The first mathematical article in this MONTHLY [*1894*, 6–11] was one on this subject by the present writer, who was co-editor of the MONTHLY from 1902 to 1908.

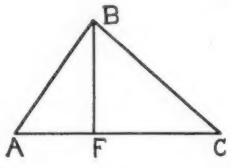
[2] The proof by Kronecker, *Vorlesungen über Zahlentheorie*, 1901, p. 35, is open to the serious objection raised by the writer in "Fallacies and misconceptions in Diophantine analysis," *Bulletin of the American Mathematical Society*, April, 1921. Moreover, it is not proved that l is an integer.

where M and N are relatively prime positive integers. By division,

$$\frac{m^2 \mp n^2}{2mn} = \frac{M^2 \mp N^2}{2MN},$$

whence, by addition, $m/n = M/N$. Since these are irreducible fractions with positive terms, we get $m = M$, $n = N$. Then $l = L$.

4. **Rational oblique triangles.** A triangle is called rational if its sides and area are expressed by rational numbers. Let ABC be a rational triangle whose sides are designated by $a = BC$, $b = AC$, $c = AB$. Since



$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$\cos A$ is rational. Hence AF is rational. Since the area equals $\frac{1}{2}BF \cdot AC$ and is rational, BF also is rational. Hence *every rational oblique triangle may be formed by juxtaposing two rational right triangles.*

By § 2, the sides of any rational right triangle are proportional to

$$2, \quad \frac{m^2 - n^2}{mn}, \quad \frac{m^2 + n^2}{mn};$$

and the sides of any second rational right triangle are proportional to

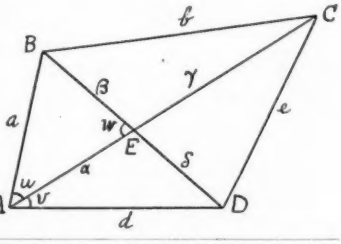
$$2, \quad \frac{M^2 - N^2}{MN}, \quad \frac{M^2 + N^2}{MN}.$$

Juxtaposing these right triangles, we see that the sides of any oblique rational triangle are proportional to

$$(3) \quad \frac{m^2 + n^2}{mn}, \quad \frac{M^2 + N^2}{MN}, \quad \frac{m^2 - n^2}{mn} \pm \frac{M^2 - N^2}{MN} = \frac{(mN \pm nM)(mM \mp nN)}{mnMN},$$

with the upper or lower signs according as the component right triangles do not or do overlap. This result was stated by Euler[1] in a posthumous fragment, but the portion of his paper which contained his proof is missing.

5. **Quadrilaterals with rational sides and diagonals.** Following Kummer,[2] we first prove that *the segments of the diagonals are rational.* By



$$b^2 = a^2 + AC^2 - 2a \cdot AC \cdot \cos u,$$

$\cos u$ is rational. Similarly, $\cos v$ and $\cos(u + v)$ are rational. Hence $\sin u \cdot \sin v$ and $\sin^2 u = 1 - \cos^2 u$ are rational. By division,

[1] *Comm. Arith. Coll.*, vol. 2, 1849, p. 648; *Opera Postuma*, vol. 1, 1862, p. 101.

[2] *Journal für reine und angewandte Mathematik*, vol. 37, 1848, pp. 1–20.

we see that $\sin u/\sin v$ is rational. Then by

$$\frac{a}{\beta}=\frac{\sin w}{\sin u}, \qquad \frac{d}{\delta}=\frac{\sin w}{\sin v}, \qquad \frac{\beta}{\delta}=\frac{a}{d}\cdot\frac{\sin u}{\sin v},$$

β/δ is rational. Thus $1+\beta/\delta = BD/\delta$ is rational. Thus δ and β are rational. Similarly, the segments α and γ of the other diagonal are rational.

The next step is much simpler than that by Kummer, who separated two cases and defined ξ algebraically, but not trigonometrically as here. *The ratios of the sides of any triangle ABE are rational if and only if*

$$c=\cos w, \qquad \xi=\frac{\sin w}{\sin u}(1+\cos u) \tag{4}$$

are rational. These are necessary conditions for the rationality of the ratios of the sides in view of the law of cosines and $\sin w/\sin u = a/\beta$. They are sufficient conditions since

$$\frac{1}{\xi}=\frac{1-\cos u}{\sin w\sin u}, \qquad \xi+\frac{\sin^2 w}{\xi}=\frac{2\sin w}{\sin u}=\frac{2a}{\beta}, \qquad \xi-\frac{\sin^2 w}{\xi}=\frac{2\sin w}{\sin u}\cdot\cos u,$$

$$\alpha=a\cos u+\beta\cos w, \qquad \frac{\alpha}{\beta}=\frac{a}{\beta}\cos u+c,$$

so that

$$\frac{a}{\beta}=\frac{1}{2}\left(\xi+\frac{1-c^2}{\xi}\right), \qquad \frac{\alpha}{\beta}=\frac{1}{2}\left(\xi-\frac{\sin^2 w}{\xi}\right)+c=\frac{(\xi+c)^2-1}{2\xi}.$$

Applying this result to the four triangles with the common vertex E, we see that there must be rational numbers ξ, η, x, y, c such that $|c|<1$ and

$$\frac{\alpha}{\beta}=\frac{(\xi+c)^2-1}{2\xi}, \quad \frac{\gamma}{\beta}=\frac{(\eta-c)^2-1}{2\eta}, \quad \frac{\delta}{\alpha}=\frac{(x-c)^2-1}{2x}, \quad \frac{\delta}{\gamma}=\frac{(y+c)^2-1}{2y}. \tag{5}$$

The product of the first and third or second and fourth left members is δ/β. Hence must

$$\frac{(\xi+c)^2-1}{2\xi}\cdot\frac{(x-c)^2-1}{2x}=\frac{(\eta-c)^2-1}{2\eta}\cdot\frac{(y+c)^2-1}{2y}. \tag{6}$$

Hence for any set of rational solutions ξ, η, x, y, c of (6) for which $|c|<1$, we obtain a quadrilateral the ratios of whose sides and diagonals are all rational, since (5) are then consistent and give rational ratios for $\alpha, \beta, \gamma, \delta$, while, as shown above,

$$\frac{a}{\beta}=\frac{\xi^2+t}{2\xi}, \quad \frac{b}{\beta}=\frac{\eta^2+t}{2\eta}, \quad \frac{e}{\gamma}=\frac{y^2+t}{2y}, \quad \frac{d}{\alpha}=\frac{x^2+t}{2x},$$

where $t=1-c^2$. Evidently we may take $\beta=1$. Thus we may assign any rational values to ξ, η, c, with $|c|<1$, and seek the rational values of x for

which the quadratic equation (6) for y has rational roots, i.e., for which its discriminant

$$\{\alpha x^2 - 2c(\alpha + \gamma)x - \alpha t\}^2 + 4t\gamma^2 x^2 \tag{7}$$

is a rational square. Only tentative methods are known for making such a quartic function of x equal to a rational square. From one such value of x others may be found by Euler's method.[1] Starting from simple values found by inspection, Kummer deduced various special rules for forming quadrilaterals with rational sides and diagonals.

6. **Rational quadrilaterals.** A quadrilateral is called rational if its sides, diagonals and area are all expressed by rational numbers. The area of $ABCD$ is

$$\tfrac{1}{2}(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha) \sin w.$$

Hence to the conditions in § 5 we must annex the condition that also $\sin w$ be rational (Kummer, *loc. cit.*). The rational solutions of $\sin^2 w + c^2 = 1$ are (§ 2)

$$\sin w = \frac{2\lambda}{\lambda^2 + 1}, \qquad c = \frac{\lambda^2 - 1}{\lambda^2 + 1},$$

where λ is rational. Hence to find all rational quadrilaterals we proceed as in § 5, but restrict c to numbers of the form $(\lambda^2 - 1)/(\lambda^2 + 1)$. For instance, we may assign arbitrary rational values to ξ, η, λ, and seek the rational numbers x for which the quartic function (7) is a rational square.

7. **Quadrilaterals formed by juxtaposing four triangles.** The fact that every rational oblique triangle may be formed by juxtaposing two rational right triangles (§ 4) suggests that a similar attempt be made for quadrilaterals. Waiving the requirement of rational area, we seek quadrilaterals whose sides and diagonals shall be rational and hence (§ 5) also the segments of the diagonals. As the first component triangle AEB take one whose sides a, α, β are measured by any rational numbers satisfying the necessary inequalities. Then

$$a^2 = \alpha^2 + \beta^2 - 2\alpha\beta c, \qquad c = \cos w, \tag{8}$$

determines c rationally. In the second component triangle BEC, we have given side β and angle $180^\circ - w$, and seek rational values of γ and b such that

$$b^2 = \beta^2 + \gamma^2 + 2\beta\gamma c. \tag{9}$$

In view of (8), we know the solution $b = a$, $\gamma = -\alpha$. Hence we set

$$\gamma = -\alpha + z, \qquad b = a + kz, \qquad z \neq 0,$$

where z and k are to be found rationally. Then (9) reduces by means of (8) to $(k^2 - 1)z = 2\beta c - 2\alpha - 2ak$. If $k^2 = 1$, either $a + z = b$ or $b + z = a$, whereas a, b, and $z = \alpha + \gamma = AC$ are sides of a triangle. Hence

$$z = \frac{2\beta c - 2\alpha - 2ak}{k^2 - 1}, \tag{10}$$

[1] L. E. Dickson, *History of the Theory of Numbers*, vol. 2, 1920, p. 639 *seq.*

where k is an arbitrary rational number distinct from ± 1. In the third component triangle CED, we have given side γ and angle w, and seek rational solutions e and δ of

$$e^2 = \gamma^2 + \delta^2 - 2\gamma\delta c.$$

For $\delta = -\beta + x$, $e = b + lx$, this reduces by means of (9) to

$$x = \frac{-2\gamma c - 2\beta - 2bl}{l^2 - 1}. \tag{11}$$

The fourth component triangle AED has two given sides α and δ and the included angle $180° - w$. The condition for rational closure is that $\alpha^2 + \delta^2 + 2\alpha\delta c$ be the square of a rational number d. Replacing δ by $x - \beta$, we get

$$X^2 + t\alpha^2 = d^2, \qquad X = x + \alpha c - \beta, \qquad t = 1 - c^2.$$

The complete solution in rational numbers, found by writing $d = X + m$, is

$$X = \frac{t\alpha^2 - m^2}{2m}, \qquad d = \frac{t\alpha^2 + m^2}{2m},$$

where m is a rational number $\neq 0$. Comparing the resulting value of x with its former value (11), we obtain a single condition on our rational parameters a, α, β, k, l, m, in terms of which all the remaining quantities are expressible rationally. While the present method is more natural than Kummer's and explains intuitively why any method must involve a rational condition of closure [(6) in Kummer's method], his method has the advantage of the symmetry due to his simultaneous consideration of the four component triangles.

8. **Parallelograms with rational sides and diagonals.** The component triangles are equal in pairs. To apply Kummer's method we need consider only the first two fractions (5), which must be equal since $\gamma = \alpha$. Hence shall

$$f \equiv \xi - \frac{t}{\xi} + 4c = \eta - \frac{t}{\eta}, \qquad \eta^2 - f\eta - t = 0,$$

$$\xi^2(2\eta - f)^2 = \xi^2 f^2 + 4t\xi^2 = (\xi^2 + 4c\xi - t)^2 + 4t\xi^2.$$

This final sum, which is therefore to be a rational square, may be deduced directly from (7) by removing the factor $\alpha^2 = \gamma^2$, and writing ξ for $-x$.

We may avoid this difficult problem of making a quartic function equal to a rational square by proceeding as in § 7, where we now require only equations (8) and (9). As before, (8) merely serves to determine c. When this value of c is inserted into (9) with $\gamma = \alpha$, we obtain the same result as if we added (8) to (9):

$$a^2 + b^2 = 2(\alpha^2 + \beta^2). \tag{12}$$

Set $\alpha + \beta = h$, $\alpha - \beta = g$. Then (12) becomes

$$a^2 + b^2 = g^2 + h^2. \tag{13}$$

To solve this in integers, set $a + g = mq$, $h + b = nq$, where m and n are relatively prime. Then $a - g = np$, $h - b = mp$. Thus

$$(14) \quad a = \tfrac{1}{2}(mq + np), \quad g = \tfrac{1}{2}(mq - np), \quad h = \tfrac{1}{2}(nq + mp), \quad b = \tfrac{1}{2}(nq - mp).$$

Since m and n are relative prime, these four numbers are integers only in the following cases: mn odd, p and q both even or both odd; just one of m and n even, p and q even. *All integral solutions of* (13) *are given by* (14) *in which* m *and* n *are relatively prime integers, while* p *and* q *are both even or* p, q, m, n *are all odd.*

The last problem is evidently the same as that of finding all triangles whose three sides and one median are rational.

9. **Conclusion.** All of the problems mentioned in this paper have been completely solved except that of a general quadrilateral whose sides and diagonals (and area) are rational. That question reduces to the problem of making a quartic function equal to a rational square. To this same problem may be reduced the solution of various questions[1] relating to triangles and quadrilaterals, as well as many questions in Diophantine analysis. A complete solution of this common outstanding problem is much to be desired.

THE TRIANGLE OF REFERENCE IN ELEMENTARY ANALYTIC GEOMETRY.

By LENNIE PHOEBE COPELAND, Wellesley College.

While the use of the triangle of reference and homogeneous coördinates is common in advanced work in mathematics, comparatively little is done along this line by undergraduates. It seems possible that younger students might find it profitable and interesting to note the behavior and shape, especially at infinity, of some of the well-known curves, when plotted on a triangle of reference. This triangle may be explained very simply,[2] since it is formed by the three lines of reference (Fig. 1) CA, CB and AB or $y = 0$, $x = 0$ and $z = 0$ corresponding respectively to the X and Y axes of the Cartesian system and the "conventional line at infinity." The distances from these lines, numbers proportional to them, or to arbitrary multiples of them, determine the coördinates of any point. The selection of the negative and positive sides of the lines may be made arbitrarily. However, it is generally more convenient to determine them in such a manner that the coördinates of points within the triangle shall be positive. This region

[1] Dickson, *History of the Theory of Numbers*, vol. 2, 1920, pp. 165–224, 497.

[2] G. Salmon, *Treatise on Conic Sections*, fifth edition, London, 1869, chapter 4.

G. Salmon, *Higher Plane Curves*, second edition, Dublin, 1873, chapter 1.

C. A. Scott, *Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry*, London, 1894.

Clebsch-Lindemann, *Vorlesungen über Geometrie*, 2. Auflage, vol. 1, part 1, Leipzig, 1906, p. 119.

O. Veblen and J. W. Young, *Projective Geometry*, vol. 1, Boston, 1910, p. 174.

corresponds to the first quadrant determined by the Cartesian axes, those portions of the plane outside the triangle corresponding in pairs to the second, third and fourth quadrants. The coördinates of the vertices are $A(1, 0, 0)$, $B(0, 1, 0)$ and C $(0, 0, 1)$. The points of CB are characterized by the fact that x equals zero, and therefore any point will have the coördinates $(0, y, z)$; similarly points on CA and AB have coördinates $(x, 0, z)$ and $(x, y, 0)$ respectively. In general if P is any point not on a side of the triangle, there exist three numbers x, y, z all different from zero such that the projections of P from the vertices on the opposite sides have the coördinates $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$. These numbers are known as the trilinear or homogeneous coördinates of P. When the coördinates are taken proportional to the distances themselves of a point from the sides of the triangle (the arbitrary multipliers equal) the bisectors of the angles of the triangle are the lines $y - z = 0$, $x - z = 0$ and $x - y = 0$, having the relation of harmonic conjugates to the corresponding bisectors of the exterior angles, the lines $y + z = 0$, $x + z = 0$ and $x + y = 0$. The former intersect the opposite sides of the triangle in the points D $(0, 1, 1)$, E $(1, 0, 1)$ and F $(1, 1, 0)$ respectively and meet in the point $P(1, 1, 1)$; the latter intersect the opposite sides produced in three points which lie on the line $x + y + z = 0$. In general any line through C has for its equation $y \pm mx = 0$ and is the harmonic conjugate of the line $y \mp mx = 0$. Similar equations hold for lines through B and A.

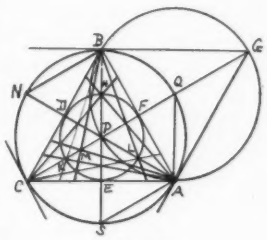


FIG. 1.

The line $x + y + z = 0$ becomes the line at infinity when the triangle of reference is equilateral. Hence it follows in this case that any curve[1] which cuts or touches the line $x + y + 1 = 0$ when referred to Cartesian axes will have infinite branches or points, when referred to this triangle. All lines intersecting on it then become parallel. Thus $NB : 2x + z = 0$ and $SA : 2y + z = 0$ are both parallel to $CF : x - y = 0$. Then if M is the midpoint of CF, the lines BM and AM are the harmonic conjugates of NB and SA or $2x - z = 0$ and $2y - z = 0$. In a similar manner the equations of other lines may be determined. With these facts in mind it is an easy matter to plot the ordinary curves, and although they appear somewhat disguised and distorted it is interesting to examine them from this new point of view. Moreover it is fascinating to see apparently simple functions don unsuspected singularities, while others break in two at unexpected places or appear with their separate branches united. For simplicity we may consider the coördinates as proportional to the distances themselves and the triangle of reference equilateral in the following examples.

The parabola $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$ which lies wholly in the first quadrant and is tangent to the x and y axes becomes the inscribed circle.

[1] The term curve as here used means the graph of an equation. Strictly speaking a given equation represents one curve when plotted on a certain set of axes and a different curve when plotted on others. Also a given curve when referred to different systems of axes does not change although its equation does.

For, writing the equation in homogeneous coördinates

$$x^2 - 2xy + y^2 - 2xz - 2yz + z^2 = 0,$$

it is seen that if $x = 0$, then $(y - z)^2 = 0$, and therefore the line $x = 0$ meets the curve twice at its point of intersection with the line $y - z = 0$, or in other words (Fig. 1) CB is tangent to the curve at D. Likewise CA and AB are shown to be tangents at the points E and F respectively. Further if $x = z$, then $y(y - 4z) = 0$, and therefore the line $x - z = 0$ cuts the curve in two points, namely the points where $x - z = 0$ intersects the lines $y = 0$ and $y - 4z = 0$. Hence BE cuts the curve at E and H, likewise CF cuts at K and F, and AD at D and L. By similar methods an infinite number of pairs of corresponding points within the triangle can be determined. Now the lines $x + my = 0$, $x + mz = 0$ and $y + mz = 0$ (when m is positive) all intersect the curve in imaginary points. Therefore no part of it lies without the triangle, and the resulting curve can be proved a circle. Here then we see that the two finite tangents $x = 0$ and $y = 0$ remain such, but the tangent at infinity having been brought into the plane is also visible.

The circumscribed circle is obtained by plotting the hyperbola $xy + x + y = 0$ which has a tangent at the origin $x + y = 0$ and the asymptotes $x + 1 = 0$ and $y + 1 = 0$. From its homogeneous equation it is seen that if $x = 0$, then $yz = 0$, or CB cuts the curve at C and B, and similarly CA cuts at C and A, AB at A and B. If $x = z$, then $z(2y + z) = 0$. Hence BE cuts at B and S, and similarly AD at A and N, CF at C and Q. The lines $x + y = 0$, $x + z = 0$ and $y + z = 0$ are tangents at the points C, B and A respectively. Thus the finite tangent is preserved, the two asymptotes become as was to be expected tangents at the points of intersection with $z = 0$, the line at infinity, and the two separate branches of the given hyperbola unite in an unbroken curve, namely a circle.

If we consider the hyperbola $xy = 1$, we see (Fig. 1) that the asymptote $x = 0$ is tangent to the curve at B when plotted on the triangle of reference. Likewise $y = 0$, the second asymptote, becomes a visible tangent at A. $x - y = 0$ cuts the curve at its points of intersection with $x - z = 0$ and $x + z = 0$, that is at P (1, 1, 1) and G (1, 1, -1). Other points obtained in a similar manner show the resulting curve to be a circle, the portion APB corresponding to that portion of the hyperbola in the first quadrant and AGB to that in the third. It is interesting to note that the parabolas $y^2 = x$ and $x^2 = y$ when referred to this triangle become equal circles through P having CA and CB, respectively, for chords.

Now let us consider the hyperbola $4xy = 1$ which is tangent to the line $x + y + 1 = 0$. From its homogeneous equation we observe (Fig. 2) that $x = 0$ and $y = 0$ are tangents to the curve at B and A respectively, while $z = 0$ cuts it at these same points; $x + y = 0$ in imaginary points; $x - y = 0$ at the points where it intersects the lines $2y - z = 0$ and $2y + z = 0$, namely M and infinity; $x - z = 0$ at the points where it intersects the lines $z = 0$ and $4y - z = 0$, namely B and T; likewise $y - z = 0$ at A and R; and in general if m is positive

$x + my = 0$ cuts the curve in imaginary points; $x - my = 0$ in two harmonic points one inside the triangle and the other outside; $x + mz = 0$ in two points one of which is B and the other the point where $x + mz = 0$ meets $4my + z = 0$, etc. Hence the curve falling entirely within the regions corresponding to the first and third quadrants, with one point at infinity, takes the form of a parabola.

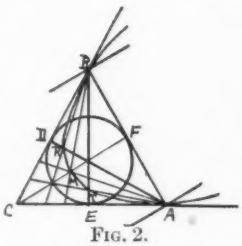


FIG. 2.

If the equation of the hyperbola is taken as $x^2 - y^2 = 1$ or $x^2 - y^2 = z^2$ then $x = 0$ cuts it in imaginary points; $y = 0$ at E and infinity; $z = 0$ at F and infinity; $y - mz = 0$ (m positive) at its points of intersection with the lines $x + z\sqrt{m^2 + 1} = 0$ and $x - z\sqrt{m^2 + 1} = 0$ giving an infinity of points inside the triangle and a corresponding infinity on the other side of the line CB, these points corresponding to those in the first and second quadrants respectively. The lines $y + mz = 0$ also cut the curve in a set of points outside the triangle in the regions corresponding to the third and fourth quadrants. The tangents of the original hyperbola $x - z = 0$ and $x + z = 0$ become a tangent at E and an asymptote respectively, while the former asymptotes $x - y = 0$ and $x + y = 0$ become one a tangent and the other an asymptote. Hence the resulting curve is a hyperbola united at one of the points where the original hyperbola broke, and breaking at one new point, namely where the line $x + y + 1 = 0$ cuts $x^2 - y^2 - 1 = 0$.

Similarly a circle about the origin in the ordinary system of coordinates referred to this triangle becomes a parabola, ellipse or hyperbola according as its radius equals, is less than or greater than, $\frac{1}{2}\sqrt{2}$.

If we refer the cubical parabola to an equilateral triangle of reference, we have in homogeneous coördinates $yz^2 - x^3 = 0$ (Fig. 3). Hence the line $x - z = 0$ cuts the curve twice at B and once at P; $x + z = 0$ twice at B and once at G: $x - y = 0$ at C, P and G: $x + y = 0$ at C; $y - z = 0$ at P; $y + z = 0$ at G; $x = 0$ twice at B and once at C; $y = 0$ three times at C, the point of inflexion. Likewise $z = 0$ cuts the curve three times at B, and as this is the only one of the lines through B ($z \pm mx = 0$) which has three intersections with the curve at this point, we see that the multiple point at B must be a cusp and the line at infinity $z = 0$ a cuspidal tangent.

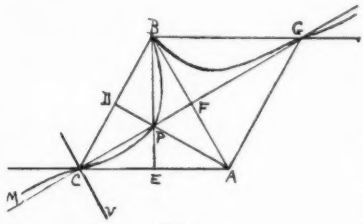


FIG. 3.

Other cubics and quartics such as the Witch of Agnesi, the Cissoid of Diocles, the Conchoid of Nicomedes and the tricuspidal quartic are interesting to plot on the triangle of reference. Moreover by varying the form of the triangle or the parameters of any given equation it may be shown that the resulting graph may assume any one of its projective forms. Thus metrical properties are changed but projective properties are preserved, and hence, in any case, the behavior at infinity may be noted.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

8. Francœur Describes a King.

Francœur, whose chief lines of interest lay in the fields of applied mathematics, was born in 1773 and died in 1849. He was on the staff of the École polytechnique and the Lycée Charlemagne and was a member of the Académie des sciences. In 1824 he went to Aix-les-Bains for the cure, and during the same period Charles Felix, the king of Sardinia from 1821 to 1831, was also taking the waters. Francœur was not one of the greatest mathematicians of France, but he was much greater as a mathematician than Charles Felix was as a king. Among the letters which Francœur wrote to his wife, who was spending the summer near Paris, is a long one in which he gossips about the events of the day and gives his impressions of the royal family. It would be interesting to know the impression, if any, that Francœur made upon the king; at any rate Francœur made considerable impression on the world, while Charles Felix made none.

Portions of the letter are as follows:

Aix in Savoy, Thursday, 29 July, 1824.

Since the post leaves tonight, I want to tell you some of the latest news of our city. The king has come; his arrival, announced long in advance, was awaited. They had stripped the hills near by of holms and pines for the purpose of planting them along by the houses. Between the trees were small blue flags bearing the arms of Savoy. The citizens also covered the walls with garlands of pines and with leaves. The roads were sanded anew and were very clean, and the public administration had done its work well. It is today that the king made his visit to Aix.

The king dined in a house prepared for his reception. He was accompanied by the queen and by her sister the duchess of Chablais. I have never seen persons with more uninteresting faces (plus pauvres visages); they seemed to me good enough people but without any brains. The king has a figure somewhat like that of the duke of Angoulême; his mouth gapes, showing his upper teeth and giving him a stupid appearance. His wife is, they say, forty-four years old. I should take her to be at least fifty-five, she seems so weak and ugly. Her sister is very old. The officers are young and newly chosen, or else they are very old serviteurs. This, without any exaggeration, is a fair picture of the court. During the dinner one could enter and pass about the table, but it was not in this way that I saw them. Instead, it was at a kind of assembly which was organized and which the king visited. All the ladies and gentlemen who take the waters gathered in the hall and he made a tour of the room, addressing remarks right and left with much politeness.

Francœur then proceeds to describe the coolness of the French to the king, remarking that

this nation is French and is unable to celebrate in behalf of a sovereign [of a territory] from which it has been separated for twenty years, and who has neither its manners nor its language.

He speaks of one of the visiting officers who, at the reception, raised his hat and cried "Vive le Roi," and of the fact that there was no response from anyone, all of which gives an interesting view of the feelings of the French people with respect to the political situation of the time. The rest of the letter is of a personal nature and has little general interest. It is not without satisfaction, however, to consider the patent fact that few people now living have ever heard of this monarch of a century ago, while the number of those who know the works of Francœur on geodesy, astronomy, and mechanics, has not lessened materially as the generations have come and gone.

9. Le Verrier and the Cost of Living.

It is consoling, in these days when the cost of living has risen far more rapidly than academic stipends, and when the family budget has assumed a new interest in university circles, to know that others, and those far greater than ourselves, have had to face the same unpleasant problem. It does not seem right that the man, or one of the two men, whose computations led to the location of a new planet without ever having seen it, should have been disturbed by the ancient *res angusta domi*, but such was the case with no less a genius than Urbain-Jean-Joseph Le Verrier (1811–1877). He was urged by Arago to devote his attention to the disturbances in the motions of the planets, and it was suggested to him that these perturbations might be caused by the presence of an undiscovered member of the solar family. He thereupon undertook the work, and in 1846 announced that his calculations showed the presence of such a planet at a specified position in the heavens, a statement at once verified by Galle, a Berlin astronomer. The discovery brought high honors to Le Verrier,—the grand cross of the Legion of Honor, a professorship in the Faculté des sciences, and the directorship of the observatory at Paris. As is well known, John Couch Adams made the same discovery independently, but Le Verrier was the first to announce it, the information being made public on September 23, 1846.

Of a considerable number of letters of Le Verrier in my collection, extending over a period of nearly thirty years, the most interesting one, from a personal standpoint, was written while he was hard at work upon his Neptune computations. It is dated five months before the discovery was announced, and is as follows:

Paris, April 18, 1846

Monsieur le Ministre,

My Father, Receveur des droits de succession at Paris, left on his death a widow, my mother, who has no income except from a pension of six hundred francs; and a daughter, Mademoiselle Léontine Le Verrier, absolutely without any money. I venture to solicit for her [the mother] a Bureau for the distribution of stamped paper at Paris. I have entire faith that my request will not fail so far as concerns the former chief under whom my Father worked, if only I have the good fortune of obtaining a favorable word from you.

My justifications in preferring this request to you, Monsieur le Ministre, are that I am connected with the giving of instruction in a school which has the honor of counting you among its former pupils, and that I have carried on certain astronomical investigations which savants have been pleased to recognize favorably, besides which I have recently been elected a member of the Academy of Sciences.

Receive, Monsieur le Ministre, the homage of my respectful regards.

U. J. Le Verrier
Member of the Institute.

His mother was then receiving a pittance of 600 francs a year, $42 at the present rate of exchange, or $120 at that time; and he, although already well known in astronomy, and a member of the Institute, found it necessary to beg for a position which could not possibly have paid much if any more. Such were some of the difficulties which faced Le Verrier in the darkness that just preceded the dawn which brought him a world-wide recognition.

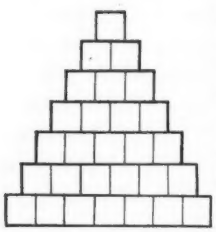
THE FORMULA $\frac{1}{2}a(a+1)$ FOR THE AREA OF AN EQUILATERAL TRIANGLE.

By G. A. MILLER, University of Illinois.

Among the interesting facts connected with the formula $\frac{1}{2}a(a+1)$ for the area of an equilateral triangle whose side is a are the following: It is remarkably inaccurate, the figures relating thereto which appear in various well known histories of mathematics are misleading, and an attempt to explain why it fails to yield correct results has been called "the first mathematical paper of the Middle Ages which deserves this name."

As regards the first of these points it may be sufficient to direct attention to the fact that in the work of Ahmes, written about 1700 B.C., the area of an isosceles triangle seems to have been found by multiplying one-half the base by one of the equal sides instead of by the altitude. This fact has been regarded as noteworthy, but when it is observed that more than two thousand years later Roman surveyors were taught to calculate the area of a special isosceles triangle by multiplying the numerical measure of half the base by a number which is even larger than the numerical measure of another side there seems to be sufficient ground for being surprised.

The present note does not aim to give a sketch of the history of this interesting formula, which sheds much light on the lack of geometric insight of the Roman surveyors during the first few centuries of the Middle Ages, appearing in such writings as Boethius's Geometry and Gerbert's Geometry. The main object of this note is to supplement statements and figures relating to this formula which appear in some of our best known modern histories of mathematics. The full benefit of the note can be secured only by those who consult the statements and figures to which references will be furnished but an effort has been made to make it instructive also to others.



The mathematics connected with the formula $\frac{1}{2}a(a+1)$ for the area of an equilateral triangle whose side is a is very elementary. If a given line segment is divided into a equal parts and if squares are arranged thereon as in the adjoining figure it is obvious that the number of these squares is the triangular number $\frac{1}{2}a(a+1)$, and that as a increases the totality of these squares approaches the area of an isosceles triangle whose altitude is equal to the base. Hence it results that if the formula $\frac{1}{2}a(a+1)$ were used to calculate the area of a given isosceles triangle whose altitude and base are both equal to a it would represent this area more and more closely as the unit of measure of the base is diminished, and the approximation could thus be made arbitrarily close by taking the unit of measure sufficiently small.

On the other hand, when the area of an equilateral triangle whose side is a

is computed by this same formula the area thus found will approach a limit, as a increases, whose ratio is to the true area as 2 is to $\sqrt{3}$. That is, the error in this case is always larger than 15 per cent.

The reason for referring here to these obvious facts is that Gerbert, who died in 1003 as Pope Sylvester II, attempted to explain in a letter to Adelbold why the area of an equilateral triangle whose side is 7 when computed by the formula $\frac{1}{2}a(a+1)$ was too large. He based his arguments on the fact that squares lying only partly within the equilateral triangle were counted according to this formula as if they were entirely within the triangle. The correctness of this part of the explanation in this particular case can be established by consulting the given figure since an equilateral triangle on the base of this figure has its vertex very slightly above the base of the uppermost square.

One of the objects of the present note is to direct attention to the fact that in the three editions of volume I of Cantor's *Vorlesungen über Geschichte der Mathematik*, 1880, 1894, 1907, pages 744, 815 and 866 respectively, the figure relating to this triangle is inaccurate. What is still more important is the fact that the corresponding figure found in various other histories is still more misleading since it represents according to the explanations in the text an isosceles triangle whose base is equal to the altitude while the text itself relates to an equilateral triangle. This fact can be verified by consulting either edition of Cajori's *History of Elementary Mathematics*, 1896 or 1917, page 132, or Günther's *Geschichte der Mathematik*, 1908, page 249. Unless the inaccuracy of these figures is noted the reader is apt to draw incorrect conclusions in regard to the merits of Gerbert's attempted explanation, which was called inaccurate by M. Chasles in his well known *Aperçu Historique*, 1875, page 506, but has been called correct in each of the mathematical histories to which reference was made above.

In view of the fact that we are now living in an age of numerous mathematical papers it may be of special interest to note here that H. Hankel[1] called Gerbert's letter to which we referred "the first mathematical paper of the Middle Ages which deserves this name." The interest which such an assertion awakens is reflected in the fact that F. Cajori quotes this statement in both editions of his *History of Mathematics*, 1894 and 1919, as well as in both editions of his *History of Elementary Mathematics*, 1896 and 1917. This high epithet of Gerbert's letter is based on the attempted explanation to which we referred and hence the appropriateness of this epithet is necessarily called in question if this explanation is regarded as inaccurate or trivial.

The triviality of the explanation results directly from the figure given above since it is obvious that the uppermost square lies above the vertex of the equilateral triangle on the same base whenever the integer $a > 7$. This triviality becomes the more obvious if it is noted that in the first part of the letter in question Gerbert speaks of a triangle whose side is 30 and his explanation would fail entirely in this case. If this triviality is granted it follows directly that the letter in question does not merit the epithet "the first mathematical paper of the Middle Ages which deserves this name."

[1] In his *Geschichte der Mathematik*, 1874, p. 314.

The present note deals admittedly with questions whose satisfactory treatment would require a large amount of space. It is hoped, however, that a few points have become perfectly clear and that these may throw light on others. In the first place, there seems to be no room for doubt in regard to the inaccuracy of the figures appearing in various well and favorably known histories of mathematics which aim to present Gerbert's explanation of the formula $\frac{1}{2}a(a+1)$ for the area of an equilateral triangle whose side is a. A correct figure appears in Bubnov's *Gerberti Opera Mathematica*, 1899, Tab. I. In the second place, it should be clear that Gerbert's attempted explanation of the formula in question exhibits too little mathematical insight and is too trivial to merit the epithet which has been so widely attributed to it. While the time from the beginning of the Middle Ages to the end of the tenth century does not exhibit much creditable work in mathematics it does present some noteworthy advances, especially in algebra and trigonometry.

NOTE ON THE PRIME DIVISORS OF THE NUMERATORS OF BERNOULLI'S NUMBERS.

By E. T. BELL, University of Washington.

1. Using the even-suffix notation for Bernoulli's numbers, $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $\cdots$ $B_{12} = -691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $\cdots$, as in Lucas, *Théorie des Nombres*, Chap. XIV, we shall prove the following

THEOREM. *If p is an odd prime which does not divide $4^r - 1$, the numerator of B_{2pr} is divisible by p.*

Hence for $r = 1$ we have a result due to John Couch Adams:[1]

COROLLARY. *If $p > 3$ is a prime, the numerator of B_{2p} is divisible by p.*

Both of these are useful as checks in numerical work, also they have a certain theoretical interest in some parts of arithmetic. Another observation due to Adams (quoted by Lucas, p. 435), states[2] that if p is an odd prime divisor of q and not a divisor of the denominator of B_{2q}, the numerator of B_{2q} is divisible by p. A comparison of this result and that which we shall establish shows that in numerical work one can often be applied with less labor than the other.

2. The proof depends upon the known fact that for $q > 0$ an integer,

$$I_q \equiv 2^{2q-1}(2^{2q} - 1)B_{2q}/q$$

is an integer.[3] Assume this for a moment; write $q = pr$, where p is an odd prime, and put $B_{2q} = N_{2q}/D_{2q}$, N_{2q}, D_{2q} being the numerator and denominator respectively of B_{2q}. Then

$$I_{pr} = 2^{2pr-1}(2^{2pr} - 1)N_{2pr}/prD_{2pr}.$$

---

[1] *Scientific Papers of John Couch Adams*, vol. 1, 1896, pp. xliv, 430.—EDITOR.

[2] J. C. Adams, Crelle's *Journal*, vol. 85, 1878, p. 269; *Scientific Papers*, vol. 1, p. 430.—EDITOR.

[3] Compare *Encyklopädie der Mathematischen Wissenschaften*, vol. II–1, 2–3, 1899, p. 183.—EDITOR.

By Fermat's theorem

$$(2^{2r})^p - 2^{2r} = M(p), \text{ (a multiple of } p),$$

and hence

$$(2^{2r})^p - 1 = M(p) + (2^{2r} - 1),$$

the right-hand member of which, and therefore also the left, is a multiple of p when and only when $2^{2r} - 1$ is divisible by p. Obviously p cannot be a divisor of 2^{2pr-1} (excluding the trivial case $p = 1$); and hence since I_{pr} is an integer it follows that if p is not a divisor of $2^{2r} - 1$, then p must divide N_{2pr}; which is the theorem.

3. It doubtless is easy in many ways to show that I_q is an integer. We give the following for its suggestiveness: the simple remark that the coefficients in the k-polynomials are integers, when combined with less obvious properties of the elliptic integrals than that which is used here, leads to a rich and unexplored field for the Bernoulli and Euler numbers. This is particularly the case when the symbolic calculus of Blissard[1] (and Lucas) is applied to the formulas furnished by the theory of transformation.

Indicating in the usual manner the modulus of the elliptic function sn x by k and writing sn (x, k), we have for the modulus unity sn $(x, 1)$, and it is easy to show (cf. Cayley, *Elliptic Functions*, p. 59) that sn $(x, 1) = -i \tan ix$, where $i = \sqrt{-1}$, and therefore

$$2ix \text{ sn } (ix, 1) = -2x \tan x.$$

But, as may readily be seen on expanding by Maclaurin's theorem, the coefficient of $(-1)^n x^{2n+1}/(2n+1)!$ $(n \geqq 0)$ in the development of sn (x, k) is of the form

$$s_0 + s_1k^2 + s_2k^4 + \cdots + s_nk^{2n},$$

in which $s_0, s_1, \cdots, s_n$ are positive integers, and hence their sum S is a positive integer.

On the other hand it is well-known (cf. Lucas, *loc. cit.* p. 262) that the coefficient of $(-1)^n x^{2n}/(2n)!$ $(n \geqq 0)$ in the development of $2x \tan x$ is $2^{2n}G_{2n}$, where G_{2n} is the $2n$th Genocchi number[2] defined by

$$G_{2n} = 2(1 - 2^{2n})B_{2n}.$$

Hence, equating coefficients of like powers of x in the two developments, we find

$$S = \frac{(-1)^{n+1}4^n}{n+1} G_{2n+2} = \frac{(-1)^n 2^{2n+1}(2^{2n+2} - 1)}{n+1} B_{2n+2};$$

and therefore on writing $q = n + 1$, we have, in the notation of § 2, $I_q =$ the integer $(-1)^n S$.

[1] J. Blissard, *Quarterly Journal of Mathematics*, vols. 6–9, 1863–1867.—EDITOR.

[2] A. Genocchi, *Annali di Scienze Mathematische e Fisiche*, vol. 3, 1852, p. 395.—EDITOR.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTIONS.

43. Is any rapid special method known for the evaluation of the Sylvestrian determinant met with so often in elimination by the dialytic method? It would seem that there must be, both on account of its interesting shape, and of its frequent occurrence; yet no text that I am familiar with deals with this form in detail. Will some colleague who is a specialist in this field and knows the periodical literature give us some information on this matter?

44. Is there any known formula for the co-factor of the element a_{ij} in the "binomial" determinant here shown?

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ C_1^2 & 1 & 0 & \cdots & 0 & 0 \\ C_2^4 & C_1^4 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ C_{n+1}^{2n} & C_n^{2n} & C_{n-1}^{2n} & \cdots & C_1^{2n} & 1 \end{vmatrix}$$

Calling the co-factors A_{ij}, it is plain that $A_{ii} = 1$, and $A_{ij} = 0$, $(i > j)$. The co-factors of the zero elements, $(i < j)$, are undoubtedly expressible in a formula; the writer would like to know if any reader of the MONTHLY has met with such a formula anywhere.

DISCUSSIONS.

Professor MacNeish shows how the intersections of two conic sections with given foci and directrices may be found by the use of ruler and compass, in case they have a focus in common. The problem reduces to special or to limiting cases of the problem of Apollonius.

Perpetual calendars have been discussed in the MONTHLY by Roman [*1915*, 241] and Morris [*1921*, 127]. Mr. Franklin calls attention in the second discussion below to a formula obtained by Zeller enabling one to calculate directly the day of the week on which any date will fall.

I. THE INTERSECTIONS OF TWO CONIC SECTIONS WITH A COMMON FOCUS.

By H. F. MACNEISH, College of the City of New York.

The determination of the points of intersection of two conic sections having given foci and directrices is in general a problem of the fourth degree, and cannot be solved by ruler and compass. If, however, the two conics have a common focus the problem reduces to one of the second degree. The geometric solution

is here made to depend on the Problem of Apollonius[1]: To construct a circle tangent to three given circles (in particular cases the circles may reduce to points or lines).

The problem may be applied to the determination of the intersection of orbits in astronomy in the case of ellipses and parabolas; and to the determination of the location of artillery by the sound of the discharge in the case of hyperbolas.[2]

Case 1. Two Parabolas. Draw a circle passing through the common focus F of the parabolas, and tangent to the two directrices. There will be two such circles; their centers C_1, C_2 will be the intersections of the parabolas.

Proof. C_1, C_2 are both equally distant from the focus and the directrix of each parabola.

Case 2. Parabola and Ellipse. Let the common focus be F and the other focus of the ellipse E. With E as center and radius equal to m, the major axis[3] of the ellipse, describe a circle E. Construct circles each passing through F and each tangent to E and to the directrix of the parabola. Their centers C_1, C_2 will be the intersections desired.

Proof. C_1 and C_2 lie on the parabola, since they are equidistant from F and the directrix. Since the line EC_1 passes through T_1, the point of tangency of circles E and C_1, $EC_1 + C_1F = EC_1 + C_1T_1 = ET_1 = m$; therefore C_1 lies on the ellipse; a similar proof holds for C_2. There will be either two points of intersection which may coincide or none.

Case 3. Parabola and Hyperbola. Let the common focus be F and the other focus of the hyperbola H. With H as center and radius equal to t, the transverse axis[3] of the hyperbola, describe a circle H. Construct circles each passing through F and each tangent to H and to the directrix of the parabola. Their centers will be the intersections desired.

Proof. Similar to the preceding case. There will in general be four intersections.

Case 4. Two Ellipses. Let F be the common focus, E and E' the other foci, m and m' the corresponding major axes. With centers E and E' and radii m and m' respectively draw circles E and E'. The point F is interior to both circles. Construct circles through F tangent to E and E'. Their centers are the intersections of the ellipses.

Proof. Similar to case 2. There are in general two intersections.

Case 5. Ellipse and Hyperbola. Let F be the common focus, E the other focus and m the major axis of the ellipse, H the other focus and t the transverse axis of the hyperbola. Draw a circle E with center E and radius m, and a circle H with center H and radius t. Construct circles through F tangent to H and E. Their centers will be the intersections of the conics.

Proof. Let one of the circles found, C_1, with center C_1 be tangent to E at T_1

[1] Solutions of some of the cases are indicated in *Methods and theories for the solution of problems of geometrical construction*, by J. Petersen, Copenhagen, 1879, problems 181, 187, 238, 276, 401, 402, 403.

[2] See H. F. MacNeish, *School Science and Mathematics*, October, 1918, p. 626. Cf. *1921*, 36.

[3] This length may easily be constructed, since foci and directrices are given.

and to H at T_1'. Then

$$EC_1 + C_1F = EC_1 + C_1T_1 = ET_1 = m,$$
$$HC_1 - C_1F = HC_1 - C_1T_1' = HT_1' = t;$$

and C_1 lies on both curves. There are in general four such intersections.

Case 6. Two Hyperbolas. The construction and proof in this case are entirely similar to those of Cases 4 and 5. There are in general four intersections.

II. An Arithmetical Perpetual Calendar.

By Philip Franklin, Princeton University.

In connection with the discussion of perpetual calendars given by Doctor Morris in the Monthly, *1921*, 127, attention is called to a formula given by Christopher Zeller[1] which enables one to obtain by merely arithmetical operations the day of the week on which any given date falls. It is thus an arithmetical perpetual calendar, giving the same information as the mechanical ones described by Doctor Morris.

$$w = \left[\frac{c}{4}\right] - 2c + \left[\frac{y}{4}\right] + y + \left[\frac{(m+1)26}{10}\right] + d,$$

where c is the number of the century,
y is the number of the year in the century,
m is the number of the month,[2]
d is the day of the month,

and the number of the day of the week to be found is the remainder obtained by dividing w by 7. $[X]$ means the greatest integer in X.

E.g., for March 4, 1921, we have

$$w = \left[\frac{19}{4}\right] - 2 \times 19 + \left[\frac{21}{4}\right] + 21 + \left[\frac{(3+1)26}{10}\right] + 4$$
$$= 4 - 38 + 5 + 21 + 10 + 4 \equiv 6 \pmod 7,$$

giving the sixth day of the week, or Friday; while for February 22, 1921, we have

$$w = \left[\frac{19}{4}\right] - 2 \times 19 + \left[\frac{20}{4}\right] + 20 + \left[\frac{(14+1)26}{10}\right] + 22$$
$$= 4 - 38 + 5 + 20 + 39 + 22 \equiv 3 \pmod 7,$$

giving the third day of the week, or Tuesday.

As noted by Zeller, to prove the formula correct, we have merely to check it for one date and notice that it gives the proper changes when we increase any of the numbers on which it depends. The reader will find this statement easy to verify if he uses the facts given in the article referred to above.

---

[1] *Acta Mathematica*, vol. 9, 1887, pp. 131 f.

[2] January and February are counted as the 13th and 14th months of the preceding year.

RECENT PUBLICATIONS.

REVIEWS.

Plane Algebraic Curves. By HAROLD HILTON. Oxford, at the Clarendon Press, 1920. 8vo. 16 + 388 pages. Price 28 shillings.

Preface: "Though the theory of plane algebraic curves still attracts mathematical students, the English reader has not many suitable books at his disposal. Salmon's classic treatise supplied all that could be desired at the time of its appearance, but the last edition was published some forty years ago, and has been long out of print. It seemed therefore as if a new book on the subject might be useful, if only to bring some more recent developments within the reach of the student.

"In the preparation of this volume I have made frequent use of the books written by Salmon, Basset, Wieleitner, Teixeira, Loria, &c. But most of the contents and examples are extracted from a very large number of mathematical periodicals. With the exception of the list at the end of Ch. XX, I have not attempted to give systematic references. In fact, in a field which has attracted so many workers, it would be almost impossible to trace the steps by which particular results have reached their present form. In some cases I cannot even remember whether a result is my own or not; but Chapters IX, XI, XVII, and XVIII contain most of my own contributions to the subject. The solutions are mine for the most part, even in the case of examples derived from other authors.

"In a book dealing with so wide a subject I can hardly hope to escape the criticism that I have included just that material which happens to interest myself, and have excluded other matter of equal or greater importance. I have not seriously dealt with problems of enumeration, such as 'How many conics touch five given conics?' I have treated all curves with the same degree and singularities as forming a single type, and have not attempted to subdivide the type by considering all their possible positions relative to the line at infinity. I have not given the properties of 'special plane curves,' unless they are representative of some general type, such as, for example, Cassinian curves, into which any quartic with two unreal biflecnodes can be projected. I have not included any discussion of curves of degree n for special values of n other than 2, 3, or 4. A thorough discussion of quintic curves would be very welcome, but at present the difficulties seem insuperable. At any rate very little work has been published on their properties. The reader will doubtless detect other important omissions. But on the whole I have tried to cover the limited ground I have selected with reasonable completeness.

"No one can really master a branch of mathematics except by working at it himself. I make no apology, therefore, for the long lists of examples. The reader can select from them few or many, as he pleases. I have given hints for solution in most cases. I hope that these will be of real assistance to the student, setting him on the right track if he is in difficulties, enabling him to check the accuracy of his results, and giving him a guarantee that the examples are not of unreasonable difficulty."

Contents—Chapter I: Introductory, 1–17; II: Singular points, 18–36; III: Curve-tracing, 37–56; IV: Tangential equations and polar reciprocation, 57–68; V: Foci, 69–75; VI: Superlinear branches, 76–87; VII: Polar curves, 88–111; VIII: Plücker's numbers, 112–119; IX: Quadratic transformation, 120–136; X: The parameter, 137–160; XI: Derived curves, 161–185; XII: Intersections of curves, 185–200; XIII: Unicursal cubics, 201–213; XIV: Non-singular cubics, 214–240; XV: Cubics as Jacobians, 241–251; XVI: Use of parameter for non-singular cubics, 252–263; XVII: Unicursal quartics, 264–297; XVIII: Quartics of deficiency one or two, 298–332; XIX: Non-singular quartics, 333–349; XX: Circuits, 350–371; XXI: Corresponding ranges and pencils, 372–383; Index, 385–388.

Pioneers of Progress: Kepler. ("Men of Science" series, edited by S. Chapman.) By W. W. BRYANT. London, Society for Promoting Christian Knowledge, 1920. 62 pages + portrait frontispiece of Kepler. Cloth. Price 2 shillings. We have referred already (*1921*, 133) to the volume on Archimedes, by Sir Thomas L. Heath, in this admirable little series of biographies.

Last two paragraphs: "Kepler's fame does not rest upon his voluminous works. With his peculiar method of approaching problems there was bound to be an inordinate amount of chaff mixed with the grain, and he used no winnowing machine. His simplicity and transparent honesty induced him to include everything, in fact he seemed to glory in the number of false trails he laboriously followed. He was one who might be expected to find the proverbial 'needle in a haystack,' but unfortunately the needle was not always there. Delambre says, 'Ardent, restless, burning to distinguish himself by his discoveries he attempted everything, and having once obtained a glimpse of one, no labour was too hard for him in following or verifying it. All his attempts had not the same success, and in fact that was impossible. Those which have failed seem to us only fanciful; those which have been more fortunate appear sublime. When in search of that which really existed, he has sometimes found it; when he devoted himself to the pursuit of a chimera, he could not but fail, but even then he unfolded the same qualities, and that obstinate perseverance that must triumph over all difficulties but those that are insurmountable.' Berry, in his *Short History of Astronomy*, says 'as one reads chapter after chapter without a lucid, still less a correct, idea it is impossible to refrain from regrets that the intelligence of Kepler should have been so wasted, and it is difficult not to suspect at times that some of the valuable results which lie embedded in this great mass of tedious speculation were arrived at by a mere accident. On the other hand it must not be forgotten that such accidents have a habit of happening only to great men, and that if Kepler loved to give reins to his imagination he was equally impressed with the necessity of scrupulously comparing speculative results with observed facts, and of surrendering without demur the most beloved of his fancies if it was unable to stand this test. If Kepler had burnt three quarters of what he printed, we should in all probability have formed a higher opinion of his intellectual grasp and sobriety of judgment, but we should have lost to a great extent the impression of extraordinary enthusiasm and industry, and of almost unequalled intellectual honesty which we now get from a study of his works.'

"Professor Forbes is more enthusiastic. In his *History of Astronomy*, he refers to Kepler as 'the man whose place, as is generally agreed, would have been the most difficult to fill among all those who have contributed to the advance of astronomical knowledge,' and again *à propos* of Kepler's great book, 'it must be obvious that he had at that time some inkling of the meaning of his laws—universal gravitation. From that moment the idea of universal gravitation was in the air, and hints and guesses were thrown out by many; and in time the law of gravitation would doubtless have been discovered, though probably not by the work of one man, even if Newton had not lived. But, if Kepler had not lived, who else could have discovered his Laws?'"

Contents—Chapter I: Astronomy before Kepler, 5–12; II: Early life of Kepler, 13–18; III: Tycho Brahe, 19–27; IV: Kepler joins Tycho, 28–34; V: Kepler's laws, 35–51; VI: Closing years, 52–57; Appendix I: List of dates, 59; II: Bibliography, 60; Glossary, 61–62.

Carnegie Institution of Washington. Year Book No. 19, 1920. Published by the Institution, Washington, U. S. A., January, 1921.

In referring to the publications of the year ending October 31, 1920, the president remarks that "three may be cited by reason of their diversity in subject-matter and by reason of special contemporary interests they have aroused." He then proceeds (page 13): "Attention was invited in the report of a year ago to the publication of Volume I of Professor DICKSON'S *History of the Theory of Numbers*. Volume II (octavo, pp. xxv + 803) of this work has appeared during this year. It is devoted to what is now called 'Diophantine Analysis,' cultivated alike by the ancient, the medieval, and the modern schools of mathematicians. It is remarkable as the branch of mathematics which has the greatest number of devotees; and its history shows well how the higher developments in science are evolved, in general, out of amateurism and dilettantism. Hence the desirability of commending both these latter stages while at the same time urging individuals to linger in neither. It is especially noteworthy in the volume in question that the French statesman Fermat (1601–1665) should be one of the most prominent of the many famous names which adorn this sort of analysis. It has turned out, in fact, that he is more distinguished for his *Opera Mathematica* than for his high conduct as councillor for the parliament of Toulouse."

BENJAMIN BOSS, director of the department of Meridian Astronomy, makes report on pages 201–207, and Director G. E. HALE gives a summary of the year's work at Mount Wilson Observatory.

GEORGE SARTON makes report (pages 383–385) concerning his work in the History of Science,

July, 1919, to August, 1920, under the headings: (*a*) work in Europe; (*b*) history of science; (*c*) Leonardo studies; (*d*) history of physics in the nineteenth century; (*e*) the new humanism.

In the report on Mathematical Physics by F. R. MOULTON, reference is made to the volume on *Periodic Orbits* (*1920*, 472), and to three papers on optical subjects published in *Visual Education*, Chicago. He gives the full program of his colloquium lectures before the American Mathematical Society in 1920, and he notes that "a chapter on numerical integration of differential equations has been written for a volume which is being prepared for the Smithsonian Institution by Professor E. P. ADAMS."

FRANK MORLEY makes report for Mathematics as follows: "Professor COBLE has published his memoir on "The ten nodes of the rational sextic and of the Cayley symmetroid" (*Amer. Jour. Math.*, vol. 41, no. 4, pp. 243–265, October 1919). He has completed a memoir on Double binary forms and the closure property,[1] which develops some new points of view and many new instances.

"Two other researches of Professor Coble which are in progress may be mentioned: In the first, the modular functions of genus 3 are used to obtain a system of irrational invariants of the ternary quartic which can be identified with a similar system arising from the set of seven points in a plane. This connects the rational invariants of the quartic with the invariants of a finite collineation group in 15 variables and would indicate that the complete system of the quartic consists of not more than 17 members.

"The second research connects his discovery that the symmetroid can be transformed by Cremona transformations into only a finite number of distinct types with the fact discovered by Schottky that the symmetroid arises from the modular functions of genus 4. Cremona transformations of the symmetroid are induced by the integer linear transformations of the modular functions, when reduced modulo 2."

Tables of the Digamma and Trigamma Functions. By Eleanor Pairman. (Tracts for Computers, edited by Karl Pearson, no. 1). Cambridge University Press, 1919. 8vo. 19 pages. Price 3 shillings.

The digamma function is $d/dz \log\Gamma(1+z)$ and the trigamma function $d^2/dz^2 \log\Gamma(1+z)$. Their computation facilitates the summation of series of the form

$$S = \sum_{i=1}^{\infty} \frac{a_0 + a_1 i + a_2 i^2 + \cdots + a_{n-2} i^{n-2}}{(p_1 i + q_1)(p_2 i + q_2) \cdots (p_n i + q_n)},$$

where the a's, p's and q's are numerical quantities, and any number of pairs of factors in the denominator may be equal. In work on the torsion flexure of aeroplane propeller blades it was found necessary to sum a large number of series of this form.

"It was not discovered until the present tables were almost completed that a certain portion of the work had been already performed. Gauss (*Werke*, vol. 3, pp. 161, 162 ···) gave tables of $d[\log\Gamma(1+z)]/dz$ correct to 18 decimal places for values of z between 0 and 1, the increment in z being .01. Professor G. N. Watson (*Report of the British Association*, 1916, pp. 125, 126) gives tables of the same function correct to 13 decimal places for all integers and halves of odd integers from 0 to 100. These two valuable tables have been made use of as a check on the present 8-figure tables. So far as has been discovered there are no other tables of $d^2[\log\Gamma(1+z)]/dz^2$."

Prefatory note by K. Pearson: "During the course of the past five years the Department of Applied Statistics in the University of London (University College) has carried out a great deal of computing work of one kind or another bearing on special war problems of a physical character. Its members have been struck by the absence of any simple text-book for the use of computers and still more by the absence of obviously necessary auxiliary tables. The present series of *Tracts for Computers* will endeavour to fill this gap as far as it lies in our power. It will not concern itself with the higher mathematical theory, but solely with the practical difficulties of the computer, or rather such difficulties as we have met with in our own experience. The first tract

[1] A memoir by Professor Coble, entitled "Multiple binary forms with the closure property," was published in *Amer. Jour. Math.*, vol. 43, January, 1921, pp. 1–19.—EDITOR.

will be followed not only by others containing recently computed tables or by the republication of old tables at present very inaccessible, but by tracts dealing with interpolation, quadrature, mechanical integration, calculating machines, tabling machines, and bibliographies of memoirs, and of tables having special value to the practical computer. In regard to the present tract, giving the values of the digamma and trigamma functions, we should ourselves have been saved many weeks of work had it been in existence four years ago. Further, we believe it will be of help not only in many physical problems other than those we have had to deal with ourselves, but to the schoolmaster who grasps the urgent importance of teaching practical mathematics to the modern school boy. The table of logarithms is not the only table that a schoolboy should learn to handle. In most modern computing laboratories a table of logarithms is very rarely used—and when used it is generally one to 10 or 14 figures[1] where multiplications are necessary which exceed the range of the ordinary multiplying machine. Nowadays the schoolboy ought to be practised in computing, and this practice should run parallel with his algebraic work. He should be exercised in the use of tables which are not becoming obsolete like the smaller tables of logarithms. He comes at a fairly early stage to the summation of series and he is liable to regard certain series as unsummable because he has not approached them numerically, just as he unfortunately regards certain integrals as unintegrable, because he is not introduced at a quite early stage to graphical, mechanical and numerical methods of quadrature. The present tract covers a very wide class of numerically summable series, and we can conceive no better practice than the schoolmaster could provide for his pupils by teaching them to sum all such series by tabular aid. If the pupil be asked at the same time to compare the result obtained by summing directly 15 to 20 terms of the series set (using tables of logarithms if he likes!), he will have learnt during the process a good deal of the practical value of logarithms, of tests for convergency, of partial fractions, of interpolation and of the value of tabular aids to the computer. He will further have realised that 'proportional parts' are neither the sole nor necessarily adequate method of entering a table;—a belief not indeed infrequently found to dominate the post-graduate as well as the school boy mind and probably arising from the same limitation of experience—the very words 'mathematical tables' being treated as synonymous with the smaller tables of common and trigonometrical logarithms."

On the Construction of Tables and on Interpolation. Part I: Uni-variate Tables; Part II: Bi-variate Tables. By Karl Pearson. (Tracts for Computers, edited by K. Pearson, nos. 2, 3). Cambridge University Press, 1920. 8vo. 54 + 70 pages. Price $3\frac{3}{4} + 3\frac{3}{4}$ shillings.

Prefatory note: "These tracts do not profess to be a complete treatise on the construction of mathematical tables, still less a full mathematical treatment of interpolation. They put together some of the practical processes, which have been found of service in the Biometric Laboratory, and state some of the difficulties which have arisen in very heavy recent computations and I would draw the attention of the pure mathematician to the necessity for their solution. As far as I am aware, but I have not made a wide search of the literature, the bi-variate central difference formulae provided are novel. They are those which naturally arise, however, when we come to deal with tables of double entry in practice.

"The main doctrine insisted on is that in ordinary mathematical tables accuracy would be gained if the tabulation of first differences were replaced by the tabulation of first central differences, and that in bi-variate tables the tabulation of the two first central differences of both variates is in the bulk of cases the sole method by which the material can be reduced within the bounds of possible publication."

On pages 62–70 of Tract no. 2, there is an annotated list of fifty works and memoirs dealing with interpolation from 1624 down to the present.

Tables of the Logarithms of the Complete Γ*-function to twelve Figures.* (Tracts for Computers, edited by Karl Pearson, no. 4.) Cambridge University Press. 4to. 1921. 14 pages + portrait frontispiece of Legendre. Price $3\frac{3}{4}$ shillings.

[1] "And where in seeking the antilog. the school boy's knowledge of the process is idle!"

This table of log Γa, from $a = 1.000$ to $a = 2.000$ is a facsimile reproduction of the one given on pages 490–499, volume 2, of Legendre's *Traité des Fonctions Elliptiques*, Paris, 1825. This table was also reproduced in O. Schlömilch's *Analytische Studien*, 1848, p. 183f. A seven-figure abridgement is given in *Smithsonian Physical Tables*, seventh revised edition, 1920, pp. 62–63. A six-figure abridgment is given in B. Williamson, *Integral Calculus*,[1] 1884, p. 169. A four-figure table for $a = 1.01$ to 2.00 is given in B. O. Peirce, *A Short Table of Integrals*, 1899. There is a very brief table, for $a = 1.0$ to 1.9, on page 30 of E. Janke and F. Emde's *Funktionentafeln mit Formeln und Kurven*, 1909. A ten-figure table for $a = 1.005$ to 2.000 for intervals 0.005 is given by G. N. Watson in *Report of the . . . British Association . . . 1916*, pp. 123–124. But earlier than any of these was a table to twenty figures given in 1813 by Gauss[2] (*Werke*, vol. 3, pp. 161, 162; vol. 10_1, p. 375), for $a = 1.00$ to 2.00. Legendre's table is the only one of these referred to in the pamphlet under review.

A seven-figure table was given on pages 58–61 of *Tables for Statisticians and Biometricians* edited by K. Pearson (Cambridge University Press, 1914). It has been found however that for many purposes especially in the construction of tables of other functions, it was needful to work with at least ten figures.

A ten-place table of $10 + \int_0^x \log_{10} \Gamma(1 + t)dt$, for $x = .01$ to 1.00 for intervals 0.01, was given by G. N. Watson (l.c., p. 124).

R. C. ARCHIBALD.

Specimen Answers of College Candidates in Plane Geometry written at the Examinations in June, 1920. (Document no. 99, April 1, 1921), New York, College Entrance Examination Board, 1921. 8vo. 22 pages. Price 25 cents.

Preface: "The following specimen answers, with the accompanying general suggestions to candidates, have been prepared for publication under the editorship of the chief reader in plane geometry, with the co-operation of the other readers. The editor desires to acknowledge hereby his appreciation of the indispensable assistance of his colleagues, at the same time accepting personal responsibility for such numerous imperfections as have doubtless resulted from his failure to give full and exact expression to their convictions.

DUNHAM JACKSON."

On pages 3–6 there is a general introductory commentary: on page 7 the paper set; and on pages 8–22, three answer books, one marked 80, another 60, and the third 55. The marks for each question and the reasons therefor are indicated.

Suggestions for Students of Mathematics. Mathematics and Life Activities. (Brown University, Bulletin of the Department of Mathematics, Number one). Providence, R. I., March, 1921. 8vo. 8 pages.

Foreword: "This Bulletin is intended primarily for students taking an introductory mathematical course in college.

"A second Bulletin will set forth the facilities and opportunities offered at Brown for pursuing the study of mathematics—especially for its own sake. This will include details regarding the

[1] This table is given in C. B. Davenport, *Statistical Methods*, second edition, New York, 1904, pp. 126–127; and in W. P. Elderton, *Frequency Curves and Correlation*, London, 1906, pp. 166–167.

[2] The *Encyklopädie der mathematischen Wissenschaften*, vol. II–1, 2, 3, 1899, p. 170, incorrectly attributes this table to Nicolai. On the other hand, the table of digamma functions, attributed to Gauss in Tract no. 1, reviewed above, was *not* by him, but computed by Nicolai under Gauss's direction.

contents of courses given in the University for the training of teachers in colleges and secondary schools."

Contents—"I: What benefits should be derived from the study of mathematics? II: Suggestions as to methods of studying mathematics. III: Mathematics and activities subsequent to college years—A. Occupations for which concentration in mathematics is desirable; B. Occupations for which concentration in mathematics combined with other subjects is desirable; C. Fields of work in which mathematical training or some knowledge of mathematics is desirable. IV: Departmental Directions."

A limited number of these pamphlets are available for distribution to those interested.

The Teaching of Geometry. By ARCHIBALD HENDERSON. *The University of North Carolina Record.* Extension series no. 39, October, 1920. 49 pages. Price 50 cents.

Headings of sections: Introduction, 3–4; The aims and results of geometrical study, 5–8; The problem of instruction (Text, teacher, pupil), 9–14; Analysis versus synthesis, 15–21; The basic problems of construction, 22–27; The problem of research, 28–45; Procedure in attacking geometrical problems, 45–48; Bibliographical note, 48–49.

NOTES.

Professor A. L. Candy's article in this MONTHLY (*1920*, 195–199) entitled "A mechanism for the solution of a equation of the *n*th degree" is reproduced in abridged form, and in Spanish, in *Revista Matemática Hispano-Americana*, December, 1920, pp. 308–309.

Reference may be given to two articles in *Monatshefte für Mathematik und Physik*, volume 30, Vienna, 1920 (216 pages). One is "Papierstreifenkonstruktion einer durch konjugierte Durchmesser gegebenen Ellipse" by K. Mack (pages 103–104); the other "Die Verallgemeinerung der Feuerbachschen Sätze" by L. Klug (pages 131–152). It is pointed out that Mack's construction is essentially that given in De La Hire, *Sectiones Conicæ*, Paris, 1685, pp. 198–199.

A series of articles, by B. LEFEBURE, published in *Revue des Questions Scientifiques* has been collected and issued in book form with the title: *Notes d'histoire des mathématiques* (*Antiquité et moyen âge*) (Louvain, Société Scientifique de Bruxelles, 11 rue des Récollets, Louvain, 1920; 8vo; 154 pages). The articles dealt with numeration and the origin of our numerals, and the history of mathematics in antiquity and in the middle ages till after the contributions of Arabian science.

An elaborate volume *Cicero: a Biography* by Torsten Peterssen (University of California Press, 1920, 5 + 699 pages) was issued as one of the series "Semicentennial Publications of the University of California, 1868–1918." The following paragraph based on information in Cicero's *Tusculans* occurs on page 173:

"The grave of the great mathematician Archimedes was supposed to be in Syracuse, but the Syracusans had neglected it, did not know where it was, and even denied its existence.[1] Cicero was familiar with a description in verse of the tomb, to the effect that it contained a globe and a cylinder. In company with some prominent citizens of Syracuse, he went to a burial place by one of the gates, and there, after some search, he found a small column with a globe and cylinder. It was almost covered with weeds. When these had been removed, the verses remembered by Cicero were found on the front of the pedestal, but only the beginnings of the lines were still legible. 'To think,' exclaims Cicero, with the amateur's delight, 'that, I, an Arpinate, should find the grave of Archimedes, the most famous citizen of Syracuse, when his fellow-citizens knew nothing about it!' In a small way, the incident is indicative of a greater movement; the Greeks were yielding their places to the Romans in nearly every sphere; in the next generation, largely as a result of Cicero's literary activity, Rome would produce authors far superior to their Greek contemporaries. As for Archimedes, Cicero seems to have studied mathematics and may have had some faint understanding of his greatness."

In the *Scientific American Monthly* for March, 1921, volume 3, pages 196–198, is one of the essays submitted in competition for the prize of five thousand dollars offered by the *Scientific American* [1921, 191]. It is entitled "The quest of the absolute; an essay on modern developments in theoretical physics." The author is Dr. F. D. MURNAGHAN, associate in applied mathematics, Johns Hopkins University. The following editorial comment is added in connection with publication of the article:

"The Judges and the Einstein Editor have no hesitation in pronouncing the essay of Dr. Murnaghan, presented herewith, to be, for the man who is equipped to read it with full understanding, altogether the most illuminating of the essays submitted in the contest, if not indeed the most successful discussion of comparable length that has appeared anywhere. The Judges were agreed that Dr. Murnaghan's essay was of doubtful value before a general audience, and that in the presence of an essay such as Mr. Bolton's appeared to be it could not properly claim the prize; but it is so very good of its kind that they clung to the last moment to the possibility of its being the best, and only allowed it to be eliminated from their consideration after the most searching examinations of Mr. Bolton's work had shown that it was all it seemed to be.

"This criticism of Dr. Murnaghan's work makes it plain that it demands publication, and equally plain that the place for it is here, rather than in the *Scientific American* proper. By all means it deserves the distinction of being the first of the competing essays to appear in the *Scientific American Monthly*, and we hasten to give it this distinction.—Editor."

About a year ago (*1920*, 218) we listed the thirteen volumes of the great edition of Euler's *Opera Omnia* which had been published at that time. Two more volumes have now been issued from the press making five published since the outbreak of the war. They are : Institutiones Calculi Integralis, part 3, edited by Engel and Schlesinger; Abhandlungen aus der Integralrechnung, 2 volumes, edited by Gutzmer and Liapounoff; Abhandlungen aus der Arithmetik, 2 volumes, edited by Rudio. The volume on Algebraische Abhandlungen, edited by Rudio, Stäckel, and Krazer, is practically complete; and the volume on Artilleriewesen, edited by Scherrer is in galley proof.

[1] In William Forsyth's *Life of Marcus Tullius Cicero*, London, 1864, vol. 1, p. 51, the following lines are quoted:

"When Tully paused amidst the wreck of time
On the rude stone to trace the truth sublime;
Where at his feet, in honored dust disclosed,
The immortal sage of Syracuse reposed."

Who wrote these lines? The incident here mentioned occurred about 75 B.C. when Cicero was 32 years of age. Archimedes died 212 B.C.—EDITOR.

The Euler Commission of the Swiss Society of Naturalists issued to its subscribers, in December, 1920, a statement[1] including the following:

"It has been possible, in spite of all difficulties to prepare five volumes of the Euler edition during the past years. Being convinced however that to many subscribers at the present time it would be a great hardship to receive such a number of volumes at once, the Euler Commission permits itself to make a present of four of these volumes and to ask for the subscription price to the fifth volume only. This applies to all subscribers, not only to private persons but also to the academies and other learned societies, as well as to the libraries.

"Pray do not conclude from this action that the financial position of the Euler undertaking is brilliant. Altogether otherwise, we look to the future with grave misgivings. Not only are the costs of composition increased more than ten fold over those before the war, but we meet also earlier unsuspected difficulties on account of the low value of money in many states. A continuation of the undertaking can therefore only be possible if all of our subscribers remain faithful to us and if we are successful in finding yet others.

"We earnestly beg you therefore, to continue to retain your highly prized good will towards the great Swiss work of the Euler Edition and to help us to bring to a happy conclusion the undertaking which has been commenced."

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 43, no. 1, January (published March), 1921: "Multiple binary forms with the closure property" by A. B. Coble, 1–19; "Einstein's, theory of gravitation: Determination of the field of light signals" by E. Kasner, 20–28; "Note on Einstein's equation of an orbit" by F. Morley, 29–32; "A one-to-one representation of geodesics on a surface of negative curvature" by H. M. Morse, 33–51; "Conjugate systems with indeterminate axis curves" by E. P. Lane, 52–68.

ANNALS OF MATHEMATICS, second series, volume 22, no. 3, March, 1921: "The asymptotic expansion of the Sturm-Liouville functions" by F. H. Murray, 145–156; "On the conformal mapping of a region into a part of itself" by J. F. Ritt, 157–160; "Conjugate nets R and their transformations" by L. P. Eisenhart, 161–181; "The applications of modern theories of integration to the solution of differential equations" by T. C. Fry, 182–211.

EDUCATIONAL ADMINISTRATION AND SUPERVISION, volume 7, no. 2, Feburary, 1921: 'Subject matter courses in mathematics for the professional preparation of Junior High School teachers" by P. M. Symonds, 61–76.

L'ENSEIGNEMENT MATHÉMATIQUE, volume 21, nos. 3–4 (published December, 1920): "Sur un théorème de cinématique" by C. Cailler, 163–169; "Généralisation des coordonnés polaires. Applications" by E. Jablonski, 170–175; "Sur les systèmes de nombres bicomplexes" by L.-G. Du Pasquier, 175–183; "Développement d'une puissance quelconque, entière et positive, de cos x ou de sin x en fonction linéaire des cos et sin de multiples de x" by E. Barbette, 184–187; "Analyse indéterminée du p^{me} degré sur les sommes de puissances égales des nombres" by E. Barbette, 188–191; "Congrès international des mathématiciens. Strasbourg, 22–28 septembre 1920," 192–209; "Les travaux de la Section de Mathématiques et d'Astronomie de l'Association française pour l'Avancement des Sciences," 209–215; "Société mathématique suisse," 215–229; "Chronique," 229–231; "Notes et documents," 232–236; "Bibliographie," 236–243; "Bulletin bibliographique," 243–250.

GRINNELL REVIEW, Grinnell College, volume 16, March, 1921: "Vindicating Euclid and Newton" [review of Girolamo Saccheri's *Euclides Vindicatus*, translated by G. B. Halsted (Chicago, 1920) and of F. Cajori's *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse* (Chicago, 1919)] by R. B. McClenon, 379.

LITERARY REVIEW, published by New York Evening Post, volume 1, January 15, 1921: "The rudeness of poets" by Christopher Morley, 7 [First paragraph: "The poet who has not learned how to be rude has not learned his first duty to himself. By 'poet' I mean, of course, any imaginative creator—novelist, mathematician, editor, or a man like Herbert Hoover. And by 'rude' I mean the strict and definite limitation which, sooner or later, he must impose upon his sociable instincts. He must refuse to fritter away priceless time and energy in the random

[1] *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1921, pp. *52–53*.

genialities of the world. Friendly, well-meaning, and fumbling hands will stretch out to bind the poet's heart in the maddening packthread of Lilliput. It will always be so. Life, for most, is so empty of consecrated purpose, so full of palaver, that they cannot understand the trouble of one who carries a flame in his heart, and whose salvation depends on his strength to nourish that flare unsuffocated by crowding and scrutiny."]; "Science histories—*History of the Theory of Numbers*. Vol. I: Divisibility and primality; Vol. II: Diophantine analysis. By L. E. Dickson. Carnegie Institution of Washington" by R. D. Carmichael, p. 9 [Quotations: "There is in many quarters a growing realization of the importance of the history of science, not only to the progress of science itself but also to the general advancement of civilization. It is certain that there is no better way to foster interest in the search for and discovery of truth than by a widespread and accurate knowledge of the way in which it has been ascertained in times past and has yielded unexpected values of essential importance. Again, it is certain that there is no greater incentive and support to the arduous duties of research than a clear conception of the way in which other thinkers have met and overcome the difficulties hindering earlier progress. . . . Whatever may be our judgment as to the ultimate relative importance of the various ends to be served by a history of science, we must recognize that the purposes of general culture cannot be met until we have first brought together the detailed facts in elaborate summaries prepared for the specialist. At the present time we do not have an adequate literature in any single body of science for serving any one of the four fundamental ends of scientific history," namely, "to enrich the general culture and intellectual life of cultivated people, to help the progress of science, and hence of human betterment, by a more widespread appreciation of its problems and the services rendered by it, to enable a scientific worker quickly to orient himself in a chapter of a science so as to proceed most readily to its detailed mastery, to enable a scientific worker to ascertain with completeness what has already been attained in a given subject."]

NATURE, volume 106, January 27, 1921: "The space-time hypothesis before Minkowski" by E. H. Synge, 693 [First two paragraphs: "It is, perhaps, not generally realized that the theory of space and time, to which Minkowski was led on experimental grounds, had been formulated on general principles sixty-five years previously by Hamilton, the Irish mathematician. The point is, however, of interest, not merely as a question of priority, but for the insight it affords into the philosophic basis of the theory, as well as for the useful mathematical methods it suggests.

"It is curious, therefore, that there should be a lack of recognition that the world of Minkowski is in all points identical with the system of quaternions of Hamilton, and that the latter mathematician specifically regarded this system as a four-dimensional expression of space and time, in which space bears to time the relation which $\sqrt{-1}$ bears to unity, time being the scalar part of the quaternion."]

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF THE U. S. A., volume 6, no. 11, November, 1920: "Seminvariants of a general system of linear homogeneous differential equations" by E. B. Stouffer, 645–648; "Some new methods in interior ballistics" by A. G. Webster, 648–659; "The permanent gravitational field in the Einstein theory" by L. P. Eisenhart, 678–682; "A simplified method for the statistical interpretation of experimental data" by G. A. Linhart, 682–684.

REVUE DE L'UNIVERSITÉ DE BRUXELLES, Bruxelles, no. 4, January, 1921: "Les mathématiques dans la biologie (La coquille du nautile)" by D'A. W. Thompson, 1–19 [Lecture delivered at the university, November 26, 1920, by the distinguished professor at the University of St. Andrews (cf. *1918*, 189, 192, 238). Last three paragraphs: "On a appelé la mathématique la servante des sciences physiques, mais elle est aussi leur reine. Par le nombre, l'ordre et la position, elle nous met sur la voie de la connaissance exacte, la voie de la vérité scientifique; ces trois termes, le nombre, l'ordre et la position, nous fournissent les premières grandes lignes d'un croquis de l'univers. Par le compas et par l'équerre, par le cercle et le carré, on nous fait mieux comprendre, comme dit le vieux charpentier dans le poème de Verhaeren: 'Les lois indubitables et fécondes, qui sont la règle et la clarté du monde.'

"Les mathématiques ne sont pas seulement une science avec ses lois, elles nous fornissent une langue—et on a dit que c'est la seule langue que le physicien puisse parler. Et un grand mathématicien écossais, qui étudia le rayon de miel il y a près de deux siècles, en a tiré la leçon que la perfection de la beauté mathématique est telle, que tout ce qui est le plus beau et le plus régulier est en même temps le plus utile et le plus excellent.

"Hier soir, sur un rayon de la bibliothèque de M. Paul Héger, j'ai mis la main sur un des ouvrages d'Henri Poincaré,—et comme vous le savez bien, même après avoir maintes fois lu ses

écrits, on y trouve toujours quelque chose de frappant et de nouveau. A la première page que j'ai ouverte, il compare la réalité objective avec l'harmonie que l'intelligence humaine croit découvrir dans la nature; et en dernière analyse il arrive à la conclusion que cette harmonie, qui s'exprime par les lois mathématiques, est la seule réalité objective, la seule vérité que nous puissons atteindre. Et en ajoutant que l'harmonie universelle du monde est la source de toute beauté, Henri Poincaré, mathématicien, arrive à la même conclusion laquelle Henri Fabre, naturaliste, est parvenu—c'est-à-dire que dans le Nombre on trouve *le pourquoi et le comment des choses*, et que l'on s'imagine y voir *la clef de voûte de l'Univers.*"]

REVUE GENÉRALÉ DES SCIENCES, volume 32, January 15, 1921: "Electricité et géométrie, après les théories récentes" by L. Bloch, 5–11.

REVUE SCIENTIFIQUE, volume 59, January 22, 1921: "Sir Norman Lockyer: la découverte de l'hélium et la température des étoiles" by H. Deslandres, 51.

SCIENCE, new series, volume 53, February 18, 1921: "A brief historical consideration of the metric system" by L. C. Karpinski, 156–157; "The history of science and the American Association for the Advancement of Science" by F. Cajori, 163–164 [Last paragraph: "In the judgment of the present writer, the dignified and logical procedure for those interested in the History of Science is, therefore, to withdraw altogether from organized historical work in connection with the American Association for the Advancement of Science until such time when the council and general session will be ready to welcome them into the Association as a separate Section."]—March 4: "Human nature as a repeating factor: that thrice told tale" by W. W. Campbell, 211–212 [First sentence: "The following comments on Professor Wood's 'Thrice told tale,' *Science*, January 14, 1921, are based upon my long experience in showing celestial objects through a great telescope to tens of thousands of Saturday night visitors, and in explaining photographs of star clusters, the milky way, spiral nebulae, etc. to thousands of others."]; "Galileo and Wood" by A. G. Webster, 212–213—March 11: "Musical notation" by R. P. Baker, 235–236 [Letter; first three sentences: "While musical notation is not a matter of great scientific interest, reform presumably is. The desirability of the changes advocated by Professors Huntington and Hall [*1921*, 35, 225] may be admitted. This leaves the space available for briefly discussing the cost."]; Review by L. C. Karpinski of A. Mieli's *Gli Scienziati Italiani* (Rome, 1921), 237–238 [cf. *1921*, 173]; "The Einstein solar field and space of six dimensions" by E. Kasner, 238–239.

TEXAS MATHEMATICS TEACHERS' BULLETIN, volume 6, no. 2, February 10, 1921: "The National Committee on Mathematical Requirements" by E. R. Hedrick, 7–15; "Quantity or quality" by Mary Campbell, 16–17; "The slide rule" by A. E. Cooper, 18–26 [A photograph of a ten-inch Keuffel & Esser slide rule is on a fly-leaf. This may be cut out and pasted on pieces of thin wood.]; "Einstein's relativity and gravitation theories" by P. M. Batchelder, 27–34; "Brown Mathematical Prizes for freshmen" by H. J. Ettlinger, 35–36; "Some elementary principles of Non-Euclidean geometry" by Ethel Burch, 37–44; "Why study mathematics?" by Arnold Dresden, 45–54 [Reprinted from *School and Society*; see this MONTHLY, *1921*, 83]; "A mathematician in love" from *Boston Transcript*, 55.

ZEITSCHRIFT FUR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 51, nos. 11–12, published December 1, 1920: "Zum Mathematikunterrichte am deutschen Gymnasium" by A. Weise, 257–262; "Die Schwingungsformel der oszillatorischen Entladung im Unterricht" by K. Hahn, 262–264; "Die Schwingungsdauer der oszillierenden Entladung im Unterricht" by W. Hillers, 264–273; "Kleine Mitteilungen," 273–276; "Aufgaben-Repertorium," 276–280; "Bücherbesprechungen" and "Zeitschriftenschau," 284–288.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF CONNECTICUT COLLEGE, New London, Conn. [*1918*, 270, 460; *1920*, 28.]

The general plan of the Club is to have at each meeting two papers. In 1919–20 the first paper was biographical and the second on any topic of general

interest. The papers are planned not to exceed twenty minutes each. After the formal papers have been read and discussed, each member who during the preceding month has read an article of interest, directly or indirectly related to mathematics, is expected to report on it. This does not exclude mathematical jokes and puzzles. This closing feature has at many meetings been most valuable and stimulating, and the only difficulty that arises is in finding time to give all a chance to report. The plan for 1920–21 is quite similar.

The Club is run entirely by its undergraduate officers. The average attendance is about 15.

The programs for 1919–20 were as follows:

October 13, 1919: Social meeting of old and prospective members at the home of Professor David D. Leib.

November 3: "Euclid" by Margaret Paul '21. Discussion of Greek mathematics. "Mathematics requirements in several women's colleges" by Louise Avery '21.

December 1: "Not 10 but 12" by Justine McGowan '20; "Descartes" by Augusta O'Sullivan '22.

January 12, 1920: "Applied mathematics" by Miss Elizabeth C. Wright, bursar of the College; "Russian peasant method of multiplication" by Barbara Ashenden '21; "Fermat" by Ruth Wells '23.

March 1: "Newton and Leibnitz" by Marcia Langley '23; "How x came to be used for the unknown quantity" by Verna Kelsey '23.

April 26: "Modern American mathematicians" by Elizabeth Hall '22; "Codes and ciphers" by Dorothy S. Wheeler '22.

May 26: Business meeting; election of officers for ensuing year.

June 1: Picnic.

The officers for 1920–21 are: President, Dorothy Pryde '21; secretary, Dorothy Wheeler '22; treasurer, Marcia Langley '23; chairman program committee, Ruth E. Wells '23.

The programs for 1920–21 are as follows:

October 8, 1920: Social meeting of old and prospective members at the home of Professor Leib.

November 8: "Mathematical ability tests on high-school students" by Augusta O'Sullivan '22; "Some impossible problems" by Barbara Ashenden '21.

December 6: "Alice in Wonderland in the land of Math." by Marion Lyon '21 (Reading); "The number 9" by Marcia Langley '23.

January 10, 1921: "Trisecting an angle and duplicating a cube" by Ruth Wells '23; Current articles on "Fleury, the blind calculator"; "The number 13 in Wagner's life" by Alice Boehringer '23.

February 7: Valentine party.

March 7: "Mathematical recreations" by Professor Ray E. Gilman, Brown University.

(Reported by Professor Leib.)

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Md.
[*1918*, 357; *1919*, 305.]

The club met every two weeks from December to May in 1919–20. In order to give each member an opportunity to participate, it was decided to have several short papers presented at each meeting. The papers were, with few exceptions, prepared and presented by the students. Visiting students from the science departments frequently took part in the discussion—an evidence of desirable correlation of related subjects. The programs were as follows:

1. "Theorems concerning a triangle and its related circles" by Deldee Groff '20; "The nine-point circle" by Loretta Whelan '20.
2. "A simple ruler-and-compass construction for a regular pentagon, and discussion of the figure arising therefrom" by Jean Burke '20 and Mildred Trueheart '22.
3. "Zeno's paradoxes" by Ruth Marshall '20; "The origin and early history of the conic sections" by Alice Langerhausen '20.
4. "Trisection of an angle and squaring of a circle by use of the quadratrix" by Mildred Grafflin '20.
5. "Geometrical fallacies, *e.g.* All triangles are isosceles" by Dorothy Biscoe '22 and Rose Diggs '22; "The prismatoid formula" by Virginia Gallup '21.
6. "The life and work of Newton" by Elsie Keith '21; "Calculation of the velocity of escape from the earth's attraction" by Mildred Brown '21.
7. "Applications of mathematics in statistical work at the Johns Hopkins School of hygiene" by Agnes Bacon, Wellesley College.
8. Joint meeting of the Mathematics and Science Clubs. "Einstein's theory of relativity" by Professor Florence P. Lewis.

In accordance with the custom of the club, refreshments, prepared by the speakers of the evening, were served at each meeting during the period for informal discussion. The annual picnic took place after the final examinations.

(Reported by Professor Lewis.)

THE MATHEMATICAL CLUB, Harvard University, Cambridge, Mass.
[*1918*, 196, 449; *1919*, 262; *1920*, 480.]

The officers for 1920–21 are: President, Instructor Rudolph E. Langer; secretary-treasurer, Instructor Bancroft H. Brown; Faculty adviser, Professor Oliver D. Kellogg.

The programs for 1920–21 were as follows:

October 6, 1920: "Cubics and biquadratics" by Professor William F. Osgood.
October 20: "Plane cubic curves" by Charles A. Spoerl '22.
November 3: "A neglected chapter of analytic geometry" by Professor Kellogg.
November 17: "Dynamics of the billiard ball" by Instructor Charles A. Rupp.
December 15: "'Necessary,' 'sufficient,' and 'necessary and sufficient' conditions" by Instructor Carl A. Garabedian.

January 5, 1921: "The mathematics of investment" by Dr. James S. Taylor, Massachusetts Institute of Technology.
January 19: "Mathematical education in Sweden" by Dr. Einar Hille.
March 2: "The dog and duck problem" [1] by Professor Henry P. Manning of Brown University.

(Reported by Instructor Brown.)

THE JUNIOR MATHEMATICS CLUB, University of Minnesota, Minneapolis, Minn.
[*1918*, 312; *1919*, 209.]

The Mathematical Club at the University of Minnesota meets at irregular intervals. Some of the meetings are of the club as a whole and some are meetings of the Graduate Section (g.s.) only. Recent programs are as follows:
March 22, 1920: (g.s.) "A class of orthotomic cubics" by Instructor Robert M. Mathews.
May 3: (g.s.) "Dynamic balancing" by Professor Burt L. Newkirk.
May 17: Annual banquet.
May 24: (g.s.) "Pressure in water due to wave motion" by Professor G. C. Priester.
October 25: "Mathematics at the University of Edinburgh" by Professor Raymond W. Brink.
November 8: (g.s.) "Functions of infinitely many variables" by Professor William L. Hart.
November 22: "The geometric representation of complex variables" by Instructor Gladys Gibbens.
January 17, 1921: "Non-Euclidean geometry" by Professor William H. Bussey.

(Reported by Professor Bussey.)

THE MATHEMATICAL CLUB OF ROCKFORD COLLEGE, Rockford, Ill.
[*1918*, 188, 409: *1920*, 77.]

The officers for 1920–21 are: President, Margaret Dodd '21; vice-president, Anna Foster '21; secretary-treasurer, Frances Regan '22.

The following programs were given during the year.
September, 1920: Election of officers.
October: Initiation of new members. "Mathematical conundrums and catch problems, spiced with poetry."
November: A semi-humorous treatment of one, two and three dimensional spaces.
December: (Open meeting) "The fourth dimension" by Professor Bessie I. Miller.
January, 1921: (Open meeting) "The Einstein theory" by Professor Miller.

(Reported by Miss Dodd.)

THE PENTAGRAM, University of Texas, Austin, Texas.
[*1918*, 273; *1919*, 364; *1920*, 321.]

The following is the report for the first two terms of 1920–21:

[1] Compare this MONTHLY, *1921*, 54–61, 91–97, 278–279, 281-282.—EDITOR.

Officers: President, Paul Boner Gr.; vice-president, Jessie M. Fouts '21; secretary-treasurer, Dr. Jessie M. Jacobs, instructor in mathematics; executive council, Dr. Goldie P. Horton, instructor in mathematics, and Renke G. Lubben '21.

There are twenty-seven student members and sixteen faculty members.

October 7, 1920: Election of officers.

October 21: "Non-euclidean geometry" by Professor Robert L. Moore.

November 4: "Periodic phenomena and periodic functions" by Paul Boner Gr.

November 20: Social meeting and initiation of new members at the home of Dr. Horton.

December 2: "The number e" by Mary Cook '23.

January 13, 1921: "Numbers" by Professor Milton B. Porter.

January 27: "Complex numbers" by Sophie Anderson '22; "Matrices" by Ruth Peden '22.

February 10: "The vibrating string" by Jessie M. Fouts '21; "Laplace's equation" by L. Vernon Robinson '22.

February 24: "One of the most practical applications of actuarial mathematics" by Mr. C. P. Rockwell, state actuary of Texas.

(Reported by Dr. Jacobs.)

NOTES.

At some colleges it seems best to form clubs for both mathematics and science students.

Professor Clyde S. Atchison, of Washington and Jefferson College, Washington, Pa., writes (February 4, 1921):

"One year ago two senior honor men in the department of mathematics, with the coöperation of the head of the department, initiated a movement for the organization of an honorary fraternity, limited in membership, for those upperclassmen especially interested in mathematics, physics, and chemistry. We now have a live organization, in which mathematics occupies its full place. Occasionally open meetings are held."

At Baldwin-Wallace College, Berea, O., according to Professor O. L. Dustheimer, about one third of the student body were present at a mathematical gathering early in January, 1921, at which questions were propounded, such as:

"Arrange the figures 1 to 9 inclusive, so their sum will be 100.

"John found $5.00; what was his gain per cent.?

"When is a number divisible by 9?

"If a melon 20 inches in diameter is worth 20 cents, what is one thirty inches in diameter worth?

"At 4 per cent., what would be the amount due last Christmas on $1.00 put at interest at the beginning of the Christian Era, to be compounded annually?

"If 6 cats eat 6 rats in 6 minutes, how many cats will it take to eat 100 rats in 100 minutes?"

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2900. Proposed by I. A. BARNETT, University of Saskatchewan.

AB is the diameter of a circle and Q_0 any point on the circumference; $Q_1, Q_2, Q_3, \cdots$ are the points of bisection of the arcs $AQ_0, AQ_1, AQ_2, \cdots$; to prove that the product of the chords of the circle $BQ_1, BQ_2, BQ_3, \cdots, BQ_n$ is equal to $OA^n \cdot (AQ_0/AQ_n)$, O being the center of the circle.

2901. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Given the length of the base of a variable triangle and the positions of the feet of the altitudes on the other two sides, find (*a*) the locus of the vertex opposite the base; (*b*) the locus of the foot of the altitude on the base.

2902. Proposed by C. N. MILLS, Tiffin, Ohio.

Find the locus of a point the feet of perpendiculars from which, on the sides of a triangle, lie on a straight line.

Note by the EDITORS: It is well known that the circumscribed circle of the triangle is at least part of the locus, by virtue of the theorem of William Wallace, *Mathematical Repository*, March 1799. No proof of this result is called for in this problem.

2903. Proposed by A. A. BENNETT, University of Texas.

Given the base of a triangle in position and length, the length of the median to the base, and the difference of the base angles; find a simple ruler and compass construction for the triangle.

2904. Proposed by N. P. PANDYA, Amreli, Kathiawad, India.

Pairs of tangents to a conic intersect on a fixed straight line; find the locus of the middle points of the chords of contact.

2905. Proposed by the late ARTEMAS MARTIN.

A circle, radius a, is drawn on a circular slate radius r; a second circle, radius a, is drawn on the slate so as to intersect the first. If a third circle, radius a, be drawn on the slate, what is the probability that it will intersect the other two?

2906. Proposed by ELIJAH SWIFT, University of Vermont.

Given any number of five digits, reverse the order and subtract the smaller of the two numbers thus formed from the larger. Show that if told the last three digits of this difference, we can find the first two and give a simple rule for determining them.

2907. Proposed by G. E. RAYNOR, Princeton University.

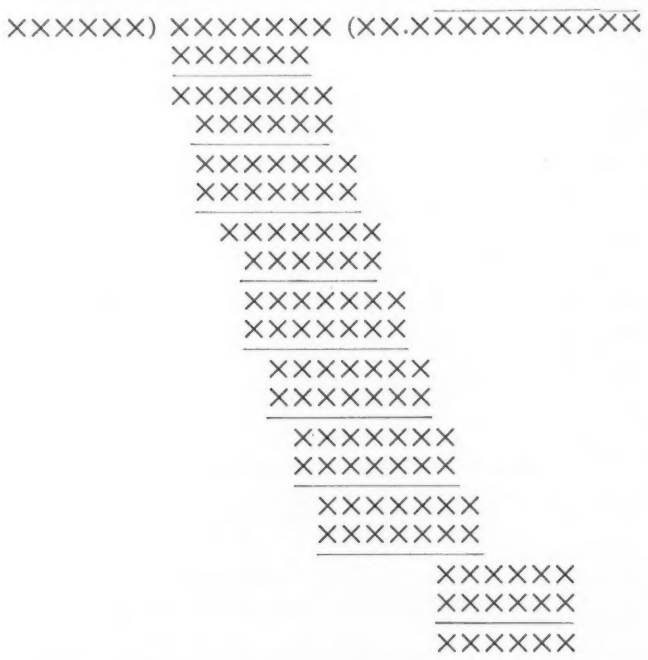
Sum the following series:

$$(a)\ \sum_{0}^{\infty} \frac{1}{(2n+1)(3n+1)}; \qquad (b)\ \sum_{0}^{\infty} \frac{1}{(3n+1)(3n+2)}.$$

PROBLEMS—NOTES.

16. Skeleton Division. This note may be regarded as a continuation of note 2 (*1921*, 37). In *Nieuw Tijdschrift voor Wiskunde*, volume 8, 1920–21, Professor F. Schuh, of Delft, proposed (page 64) two arithmetical puzzles; one of them is herewith reproduced.

"The following calculation represents the development of an irreducible fraction to a repeating decimal:



Crosses indicate digits—all of which are possible. The repeating digits are shown by a line over the crosses. What irreducible fraction has been developed?"

ARC.

17. Curves of Pursuit (Compare *1921*, 54–61, 91–97, 184–185). I see that the AMERICAN MATHEMATICAL MONTHLY has been interesting itself in early examples of the Curve of Pursuit. Perhaps some of its readers may like these notes on four examples of it which appeared in England before 1750.

The general problem of the form of the curve, the pursued travelling in a straight line, is discussed in T. Simpson's *New Treatise of Fluxions*, London, 1737, p. 170, that is five years after the appearance of Bouguer's memoir.

A problem on it, by Robert Heath, appeared in *Ladies Diary*, 1742 and 1743 (Leybourn's reprint, vol. 1, p. 323), in the following form: "Admiral Vernon

sailing on a south course, from Jamaica to Carthagena, sees Don Blass right before him steering west, along the shore. He now continually bears directly upon him in a right line; when coming up with him, it appears that the Don had sailed 8 leagues during the chace, and that the said admiral was 7 leagues distant from him when the chace began: Now, supposing each ship's motion to be uniform during the whole chace, to find from thence the distance sail'd by Admiral Vernon?"

The following problem was propounded by Thomas Perryam in *Gentleman's Diary*, 1749: "A Captain of a Privateer seeing a Merchant Ship at S.S.E. sailing due West, continually bears directly upon her in a right Line; and coming up with her, it appear'd that the Merchant's Ship sail'd 30 Miles during the Chace, and that the said Privateer was 21 Miles distant from her when the Chace first began: Now supposing each Ship's Motion to be uniform during the whole Chace, To find from thence the Distance sail'd by the said Privateer, its East and West Departure, and also their Difference of Latitude when they were North and South of each other." A solution by fluxions was given in 1750.

Another question on the subject was propounded by John Ash in *Ladies Diary*, 1748, (Leybourn's reprint, vol. 2, p. 15) in the following form: "A spider, at one corner of a semi-circular pane of glass, gave uniform and direct chase to a fly, moving uniformly along the curve before him; the fly was 30° from the spider at the first setting out, and was taken by him at the opposite corner. What is the ratio of both their uniform motions[1]?" On this question the editor said "Mr. Landen sent us a true method [of solution]; but the calculus being so operose, it was not wrought out. And no method appearing to us yet elegant enough for a place, it will be next year before we shall have time to catch the solution." No further remarks on it appear in subsequent numbers of the *Diary*.

It would seem then that before 1750 such questions had become sufficiently familiar in England to appear in popular journals.

W. W. Rouse Ball.

Trinity College, Cambridge.
March 1, 1921.

18. Radius of the sphere circumscribing a tetrahedron. A member of the Association proposed the following problem which during the past century has been frequently solved: "Find, in terms of the lengths of the edges of a tetrahedron, the radius, R, of the circumscribing sphere." Denoting the pairs of opposite edges of the tetrahedron by a, a_1; b, b_1; c, c_1, L.N.M.Carnot derived[2] in 1806 the relation

$$\begin{aligned}4R^2(a_1{}^4a^2 + a^4a_1{}^2 + b_1{}^4b^2 + b^4b_1{}^2 + c_1{}^4c^2 + c^4c_1{}^2 + a^2b_1{}^2c_1{}^2 + c^2a_1{}^2b_1{}^2 \\ + b^2a_1{}^2c_1{}^2 + a^2b^2c^2 - a^2b^2b_1{}^2 - b^2c^2b_1{}^2 - b^2a_1{}^2b_1{}^2 - a^2a_1{}^2b_1{}^2 - b^2b_1{}^2c_1{}^2 \\ - c^2b_1{}^2c_1{}^2 - a^2b^2a_1{}^2 - a^2c^2a_1{}^2 - b^2c^2c_1{}^2 - a^2c^2c_1{}^2 - a^2a_1{}^2c_1{}^2 - c^2a_1{}^2c_1{}^2) \\ + 2b^2c^2b_1{}^2c_1{}^2 + 2a^2c^2a_1{}^2c_1{}^2 + 2a^2b^2a_1{}^2b_1{}^2 - a^4a_1{}^4 - b_1{}^4b^4 - c_1{}^4c^4 = 0.\end{aligned}$$

[1] Compare the notes on problem 2801 below.—Editor.

[2] *Mémoire sur la Relation qui existe entre les distances respectives de cinq points quelconques pris dans l'espace.* Paris, 1806, p. 11.

In 1847 Brassine gave,[1] without proof, the formula:

$$R = \frac{1}{24 \cdot V}\sqrt{(aa_1+bb_1+cc_1)(aa_1+bb_1-cc_1)(aa_1+cc_1-bb_1)(cc_1+bb_1-aa_1)},$$

or

$$R = \frac{1}{6 \cdot V}\sqrt{p(p-aa_1)(p-bb_1)(p-cc_1)},$$

where V is the volume of the tetrahedron, and $2p = aa_1 + bb_1 + cc_1$. That is, six times the product of the volume of the tetrahedron and of the radius of the circumscribed sphere is numerically equal to the area of a triangle whose sides are of lengths aa_1, bb_1, cc_1. This result was published in 1821 by Crelle.[2] In 1752 Euler gave in effect, the following expression,[3] in modern notation, for V:

$$\frac{1}{12}\sqrt{\begin{aligned}&[a^2a_1^2(b^2+b_1^2+c^2+c_1^2) - a^2a_1^2(a^2+a_1^2) + b^2b_1^2(c^2+c_1^2+a^2+a_1^2)\\ &\quad - b^2b_1^2(b^2+b_1^2) + c^2c_1^2(a^2+a_1^2+b^2+b_1^2) - c^2c_1^2(c^2+c_1^2)\\ &\quad - a^2b^2c^2 - a^2b_1^2c_1^2 - b^2c_1^2a_1^2 - c^2a_1^2b_1^2].\end{aligned}}$$

Killing and Hovestadt state this relation[4] in the form

$$\begin{aligned}144V^2 = {}&(a^2+b^2+c^2+a_1^2+b_1^2+c_1^2)(a^2a_1^2+b^2b_1^2+c^2c_1^2) - 2a^2a_1^2(a^2+a_1^2)\\ &- 2b^2b_1^2(b^2+b_1^2) - 2c^2c_1^2(c^2+c_1^2) - a_1^2b^2c^2 - a^2b_1^2c^2 - a^2b^2c_1^2\\ &- a_1^2b_1^2c_1^2.\end{aligned}$$

Baltzer gave[5] the determinant form:

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 \\ 1 & a^2 & 0 & c_1^2 & b_1^2 \\ 1 & b^2 & c_1^2 & 0 & a_1^2 \\ 1 & c^2 & b_1^2 & a_1^2 & 0 \end{vmatrix},$$

and Joachimsthal, and Dostor the following:[6]

$$576V^2R^2 = -\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & c_1^2 & b_1^2 \\ b^2 & c_1^2 & 0 & a_1^2 \\ c^2 & b_1^2 & a_1^2 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & aa_1 & bb_1 & cc_1 \\ aa_1 & 0 & cc_1 & bb_1 \\ bb_1 & cc_1 & 0 & aa_1 \\ cc_1 & bb_1 & aa_1 & 0 \end{vmatrix}.$$

[1] *Nouvelles Annales de Mathematiques*, vol. 6, 1847, p. 227; a demonstration by de Perrodil is given on pages 396–398 of the same volume.

[2] A. L. Crelle, *Sammlung mathematischer Aufsätze und Bemerkungen*, Berlin, vol. 1, p. 117.

[3] *Novi comment. acad. sc. Petrop.*, vol. 4 (1752–53), 1758, p. 159.

[4] W. Killing und H. Hovestadt, *Handbuch des mathematischen Unterrichts*, Leipzig, vol. 2, 1913, p. 421.

[5] R. Baltzer, *Théorie et Applications des Déterminants*. Traduit de l'allemand. Paris, 1861 p. 206.

[6] F. Joachimsthal, Crelle's *Journal*, vol. 40, 1850, p. 33. G. Dostor, *Eléments de la Théorie des, Déterminants*, deuxième éd., Paris, 1883, pp. 281–282; also *Nouvelles Annales de Mathématiques*, vol. 32, 1873, p. 374.

Other derivations of expressions for the radius of the circumscribed sphere were given by Legendre, *Eléments de Géométrie*, 8e éd., Paris, 1809, pp. 302–304 of note V; by Baltzer, *Die Elemente der Mathematik*, Leipzig, volume 2, 1860, pp. 348–349; and by G. Holzmüller, *Elemente der Stereometrie*, Leipzig, volume 2, 1900, pp. 228–231. ARC.

PROBLEMS—SOLUTIONS.

2801 [1920, 31; 1921, 54–61, 91–97]. Proposed by A. S. HATHAWAY, Houston, Texas.

A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $k : 1$, determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

I. ADDITIONAL REMARKS BY THE PROPOSER.

If we let $d\sigma$ denote differential along the apparent path of the dog, we shall have

$$\left(\frac{d\sigma}{ds}\right)^2 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = MQ^2/a^2,$$

whence $d\sigma/dt = MQv/a$, where $v = ds/dt$ is the real velocity of the duck.

When $k > 1$, $MQ \geqq ka - a$, and so the apparent velocity of the dog within or on the edge of the pond is never less than $(k - 1)v$.

When $k = 1$, the apparent velocity of the dog is never zero within the pond, nor on the edge except at P. In fact the dog will approach P as limiting point.

When $k < 1$, MQ will be zero at the point where the ray $\sin\theta = k$ intersects the circle CKP. Let α be this value of θ and let w be the distance of Q from this point. Then

$$w^2 = r^2 + a^2\cos^2\alpha - 2ar\cos\alpha\cos(\theta - \alpha),$$

and

$$w\,dw/ds = -k(r + a\cos\alpha)[1 - \cos(\theta - \alpha)],$$

which is never positive, and is zero only when $\theta = \alpha$. Therefore w is always decreasing (except when $\theta = \alpha$), and the apparent path of the dog is a spiral converging to the point where the apparent velocity is zero, $\theta = \alpha$, $r = a\cos\alpha$.

It is interesting to note that if the speed of the dog be less than that of the duck, there is one position of starting where the relative positions of the two will remain unchanged, and that this is the limiting goal of the dog from whatever position he starts within the pond.

A number of interesting results may be deduced by determining entrances and exits on fixed curves. Thus on $r = ma\cos\theta$, exits and entrances are divided by the ray $(m - 2)\sin\theta + k = 0$, exits over the shorter arc.

II. REMARK BY H. P. MANNING, Providence, R. I.

Mr. Morley has made a slight mistake (*1921*, 60) in regard to the cusps and inflections of his integral curves. The substitutions on page 55 lead first to an irrational equation for dv/dp, and the second derivative of v for a curve of pursuit is zero only when the integral curve crosses that part of the cubic which lies to the left of the point for which $p = c$. Between the two parts of the cubic all integral curves are concave downwards, but at any point outside of the cubic they curve in opposite directions; and so from a cusp the two branches curve away in opposite directions, one less steep and the other steeper than the slope $-2c$, and the cusps are all ordinary cusps, and not of the rhamphoid type. Indeed, one such cusp is at the point (0, 1), where one branch corresponds to the curve of pursuit and the other to the "curve of flight."

III. REMARKS AND HISTORICAL NOTES BY H. P. MANNING, AND R.C. ARCHIBALD, Brown University.

We have remarked before (*1921*, 92) that the earliest reference then found, to a curve of pursuit where the pursued moved on a circle, was in Ficklin's problem of 1859. Mr. Ball has noted

above that a similar problem, that of a spider and a fly, was proposed, but not solved, in the *Ladies Diary* more than a century earlier. It appears now that this *Diary* problem suggested the following[1]: "To find the nature of the curve, described by a body giving uniform and direct chase to another, supposing the body pursued to move uniformly in the periphery of a given circle." The "solution," published in 1751, leads to Professor Hathaway's equations.

Using Professor Hathaway's notation and figure (*1921*, 94) and regarding the problem as that of a dog and duck, we can describe this "solution" as essentially the following:

First, the two fundamental equations, Professor Hathaway's (3) and (4), are derived.

The velocity with which the dog gains on the duck is equal to his own velocity diminished by the component along QP of the duck's velocity. This gives us at once equation (3).

To get equation (4) take the component along CQ of the dog's velocity. CQ being z, we have

$$dz/ds = k \sin KCQ = k(a \cos \theta - r)/z.$$

But

$$z^2 = r^2 - 2ar \cos \theta + a^2,$$

whence

$$z\, dz = (r - a \cos \theta)dr + ar \sin \theta d\theta,$$

or

$$(a \cos \theta - r)(k + dr/ds) = ar \sin \theta d\theta/ds,$$

which gives at once equation (4).

Ratios are used instead of the trigonometric functions, and x and y are used for r and PK $(= a \cos \theta)$.

The equation in x and y, equivalent to Professor Hathaway's (5), contains the expression $\sqrt{a^2 - y^2}$. It is proposed to develop this in a series and then to use "the method of resolving fluxional equations," a reference being given to Simpson's *Fluxions*. For the particular case of the spider and fly problem it is found that, velocity of the spider : velocity of the fly = 1.16 : 1.

The discussion concludes with the following note: "When the velocity of the body pursued is the greater of the two, the required curve will be a spiral converging continually nearer and nearer to the circumference of a circle concentric with the given one; whose radius is to that of the given one in the ratio of the lesser velocity to the greater."

As a further bibliographical note, reference may be given to problem 2971, proposed by the late Artemas Martin in *Educational Times*, volume 22, 1869, p. 141. It was as follows: "Show that the solution of the famous 'curve of pursuit problem' when the object pursued moves in the circumference of a circle and the pursuer starts from the center, can be made to depend upon the solutions of the differential equations

$$d\theta = -\frac{dt}{n - \cos \phi} \quad (1), \qquad tdt = r\{d(t \sin \phi) - ntd\phi\} \quad (2);$$

where r is the radius of the circle, t the distance the two objects are apart at any time during the motion, ϕ the angle t makes with a tangent to the circle, and θ the arc described by the pursued object from the commencement of the motion, supposing the pursuer to move n times as fast as the pursued." A solution of this problem, by James McMahon, appeared in *Mathematical Questions and Solutions from the "Educational Times"* volume 51, 1889, p. 159.

Captain Henri Brocard, of Bar-le-Duc, France, has kindly drawn our attention to solutions of the following problems (21, 22) proposed by Dr. W. Kapteyn in *Wiskundige Opgaven met de Oplossingen*, new series, volume 10, pp. 50–52, 1907: "21. A point C moves on a circle of radius equal to unity. Another point, situated originally at the center O of the circle, moves with the same velocity as the point C on a curve whose tangent passes constantly through this point. Prove that the radius of curvature at any point M of this curve is equal to the segment [measured from C] intercepted on the radius OC by the normal in M." 22. "Consider the radius of curvature ρ [at M], of the curve referred to in the previous question, as a function of the distance, p, of the origin from the tangent to this curve [at M]. Form the differential equation connecting ρ and p."

Of the first of these problems three solutions were published. In one of them, by the proposer, equations very similar to those employed by Mr. Morley (*1921*, 55) are derived.

The problems were reprinted in *Nouvelles Annales de Mathématiques*, February, 1907, pp. 95, 476; a solution of the first was given on page 173 by M. d'Ocagne, who refers to his paper "Centre de courbure des courbes de poursuite" in *Bulletin de la Société Mathématique de France*, volume 11, 1883, pp. 133–134.

[1] J. Turner, *Mathematical Exercises*, London, 1750–1752; no. 2, 1751, p. 32; no. 3, 1751, pp. 77–80. The problem and "solution" are reprinted in T. Leybourn's *Mathematical Questions proposed in the Ladies' Diary*, vol. 2, 1817, pp. 15–17.

2805 [1920, 32]. Proposed by C. N. MILLS, Brookings, S. Dakota.

Derive the expression for volume

$$v = \int\int\int \rho^2 \sin\phi d\rho d\phi d\theta.$$

In Byerly's *Integral Calculus*, page 183, revised edition, is a method of revolution, and in Czuber's *Integralrechnung*, page 200, is a method using the Jacobian determinant.

Required, a simple method one might use in developing the volume integral in polar coordinates.

SUGGESTION BY PAUL CAPRON, U. S. Naval Academy.

Draw a figure to show an octant of a sphere, as it might be of the earth with the North Pole atop, from Long. 0° to Long. 90° W. and from Lat. 0° to Lat. 90° N. The angle θ will correspond to the longitude, ϕ to the complement of the latitude. Show two meridians $d\theta$ apart and two parallels $d\phi$ apart. On the rectangular area so bounded build up an element of volume by extending through its corners four radii, projecting a distance $d\rho$ to a concentric spherical surface, on which they mark the corners of another rectangular base.

The sides of the inner rectangular base are $\rho \sin\phi d\theta$, $\rho d\phi$, $\rho \sin(\phi + d\phi)d\theta$, $\rho d\phi$; the sides of the outer base are the same, with $(\rho + d\rho)$ in place of ρ.

This point in the discussion should be reached at just about the end of hour. Conclude by leaving to the individual students the task of proving, each to his own satisfaction, that the neglected infinitesimals are of order higher than the first. Serve out enough applications to furnish a diversion of interest and start something useful and fascinating at the next lecture.

Also answered by T. M. BLAKSLEE, A. R. NAUER, H. L. OLSON, D. H. RICHERT, and ELIJAH SWIFT.

2806 [1920, 32]. Proposed by R. E. MORITZ, University of Washington.

An anthropologist told me recently that large numbers of Russian peasants, whose knowledge of numbers is limited to multiplication and division by 2, employ the following method of multiplication which they were taught by a priest.

(1) Write the two numbers to be multiplied in the same horizontal line.

(2) Multiply the first number by 2, and write the product under the number so multiplied.

(3) Divide the second number by 2, discarding the remainder 1 when it occurs, and write the quotient under the number so divided.

(4) Treat the product and quotient thus obtained in the same manner as the original numbers. Continue this process until the quotient 1 is obtained.

(5) Strike out all the numbers on the left for which the corresponding numbers on the right are even.

(6) Add the remaining numbers on the left. Their sum is the required product.

Problem: Prove that this rule is correct.

SOLUTION BY P. R. RIDER, Washington University.

The method depends on the fact that any number can be written in the binary scale of notation, that is, as a sum of positive integral powers of 2. To express a number in this way we divide successively by 2 until the quotient is 1. This 1 will be the first digit, beginning at the left, and the remainders (always either 1 or 0) will be the other digits expressing the number. That is, the remainder after the rth division by 2 will be the digit in the rth place, counting from the right, in the binary scale expression of the number, or the coefficient of 2^{r-1} in the expansion of the number in positive integral powers of 2. (See Todhunter's *Algebra for the use of Schools and Colleges*, chapter 29.) Thus, the multiplication process considered consists essentially in expressing one number in positive integral powers of 2 and multiplying the other number by this expression.

For instance, suppose that the numbers to be multiplied by the method are a and b, a heading the first column and b the second. Then the second column is the work of developing b in powers of 2, and the numbers in the first column are the terms of this development, each multiplied by a. For, if a number in the rth line of the second column is odd, there will be a remainder of 1, and

the corresponding digit in the expression of b in the binary scale is 1, giving the term of value $1 \times 2^{r-1}$ in the development of b in powers of 2. If, on the other hand, the number in the rth line of the second column is even there will be no remainder, and the corresponding digit in the binary scale expression of b is 0, giving the term of value $0 \times 2^{r-1}$ in the development of b. But the number in the rth line of the first column is $2^{r-1} \times a$, and since all numbers in the first column that are opposite even numbers in the second column are stricken out, the sum of the remaining numbers will be precisely $b \times a$.

A numerical example will make this much clearer. The work for the multiplication of 14 and 83 would appear (except for the figures in parentheses) as follows:

$(2^0 \times 14 =)^*$ 14	83
$(2^1 \times 14 =)^*$ 28	41
$(2^2 \times 14 =)$ 56	20
$(2^3 \times 14 =)$ 112	10
$(2^4 \times 14 =)^*224$	5
$(2^5 \times 14 =)$ 448	2
$(2^6 \times 14 =)^*896$	1
1162	

The number 83 expressed in the binary scale of notation would be 1010011 (*i.e.*, $83 = 2^1 + 2^4 + 2^1 + 2^0$). Thus the sum of those parentheses marked with an asterisk is $(2^6 + 2^4 + 2^6 + 2^0) \times 14$, or 83×14.

In presenting a similar discussion Professor U. G. MITCHELL cited the article in this MONTHLY, *1918*, 139–142, by Professor R. C. ARCHIBALD, entitled: "The binary scale of notation, a Russian peasant method of multiplication, the game of nim, and Cardan's rings." Many references are there given to the literature of the history and discussion of the binary scale and its applications.—EDITORS.

Also solved by T. M. BLAKSLEE, B. A. BERNSTEIN, PAUL CAPRON, CARL GUNDERSEN, W. H. HAYS, A. M. KENYON, THEODORE LINQDUIST, ROSCO LAMONT, H. F. MACNEISH, L. C. MATHEWSON, H. L. OLSON, ARTHUR PELLETIER, W. B. PIERCE, D. H. RICHERT, H. S. UHLER, and C. C. WYLIE.

2822 [1920, 185]. Proposed by A. M. HARDING, University of Arkansas.

Show that the sum of the series:

$$1 + 3\cdot 2 + 5\cdot 2^2 + 7\cdot 2^3 + \cdots + (2n-1)2^{n-1}$$

(to n terms) is $3 - 2^n + (n-1)2^{n+1}$.

SOLUTION BY LOUIS O'SHAUGHNESSY, Virginia Polytechnic Institute.

Set

$$S = 1 + 3\cdot 2 + 5\cdot 2^2 + 7\cdot 2^3 + \cdots + (2n-3)2^{n-2} + (2n-1)2^{n-1}.$$

Then

$$2S = 2 + 3\cdot 2^2 + 5\cdot 2^3 + \cdots + (2n-5)2^{n-2} + (2n-3)2^{n-1} + (2n-1)2^n.$$

Hence,

$$S = -1 - 2\cdot 2 - 2\cdot 2^2 - 2\cdot 2^3 - \cdots - 2\cdot 2^{n-2} - 2\cdot 2^{n-1} + (2n-1)2^n,$$

$$= -1 - \sum_{2}^{n} 2^n + (2n-1)2^n = -1 - (2^{n+1} - 4) + (2n-1)2^n = 3 - 2^n + (n-1)2^{n+1}.$$

Also solved by T. M. BLAKSLEE, H. N. CARLETON, P. J. DA CUNHA, E. B. ESCOTT, R. M. GINNINGS, H. HALPERIN, HARRY LEVY, L. C. MATHEWS, H. L. OLSON, ARTHUR PELLETIER, A. V. RICHARDSON, ETHELDRED A. WILLMOTT, and C. C. WYLIE.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

At Defiance College, O., Mr. E. H. VANCE was appointed professor of mathematics in January, 1921.

Professor EVA CHANDLER, of Wellesley College, retired in June, 1920, as professor emeritus. She was instructor of mathematics at Wellesley, 1879–1888; associate professor, 1888–1912; professor, 1912–1920.

Dr. GERTRUDE I. MCCAIN, professor of mathematics at Oxford College for Women, O., will teach at the summer school of Hunter College, New York City, in 1921.

Mr. R. L. WILDER, of Brown University, has been appointed instructor of mathematics at the University of Texas.

Mr. D. E. WHITFORD, of Brown University, has been appointed instructor of mathematics at the University of Rochester.

Instructor H. T. STETSON, of Harvard University, has been promoted to an assistant professorship.

Professor FRANK MORLEY, of the Johns Hopkins University, is now in England on leave of absence for the second semester of the current year.

At the University of Alberta, Dr. S. D. KILLAM, associate professor of applied mathematics, has been promoted to a full professorship, and Assistant Professor J. W. CAMPBELL (*1920*, 336, 498) has been promoted to an associate professorship.

W. F. PICKERING, assistant professor of astronomy and director of the Mandeville Station in Jamaica of the Harvard Astronomical Observatory, has been granted leave of absence for one year. He went first to England. Later he will probably study volcanic craters in the neighborhood of the Mediterranean Sea.

Miss RACHEL T. EASTERBROOKS, of Brown University, has been appointed instructor of mathematics and assistant in physics at Hood College, Frederick, Md.

Professor W. W. RANKIN, of the University of North Carolina, who has been on leave of absence for the last two years acting as instructor at Columbia University, has been appointed in charge of the department of mathematics at Agnes Scott College, Decatur, Ga.

Professor JOSEPH LIPKA, of Massachusetts Institute of Technology, is to spend the year 1921–1922 in Europe and Great Britain. He expects to work for some months with Professor LEVI-CIVITA in Rome.

At Union College, Mr. A. D. SNYDER of Lafayette College, Easton, Penn., and Mr. RALPH BENNETT have been appointed instructors of mathematics in place of Mr. COMSTOCK and Mr. TERWILLIGER (cf. *1921*, 234).

At Westminster College, New Wilmington, Pa., Associate Professor J. V. MCKELVEY of Iowa State College has been appointed professor of mathematics.

Associate Professor W. A. MANNING, of Stanford University, has been promoted to a professorship of mathematics.

Dr. M. C. FOSTER, of Yale University, has been made instructor of mathematics at the university.

The following appointments and promotions were made in the autumn of 1920:

At the Carnegie Institute of Technology, Pittsburgh, Mr. J. B. ROSENBACH, formerly of the University of Illinois, and Mr. C. H. KOHLER were appointed instructors of mathematics in the School of Science and Engineering. Mr. J. R. EVERETT, Dr. GEORGE HESS, and Mr. H. F. KOHL were appointed instructors of mathematics in the School of Applied Industry.

At the Georgia School of Technology, Atlanta, Ga., Mr. D. L. STAMY was promoted from an instructorship of mathematics to an assistant professorship. The following were appointed instructors of mathematics: Mr. W. W. ELLIOTT, Mr. P. L. ARMSTRONG, Mr. J. L. DRISCOLL, and Mr. L. H. HOLLER.

Miss IDA WHITAKER was appointed instructor of mathematics at the Municipal University of Akron, O.

At the Colorado School of Mines, Golden, Mr. G. W. GORRELL, formerly of DePauw University, was appointed professor of mathematics and Mr. W. R. HALE, formerly of Broadus College, was appointed assistant professor.

At Phillips University, East Enid, Okla., Mr. F. E. KNOWLES, formerly of the Oklahoma Agricultural College, was made professor of mathematics and physics. Mr. W. M. REEVES, formerly of Cotner College, was appointed associate professor of mathematics. Mr. A. J. HARGETT resigned to become professor of mathematics in Texas Christian University.

At Henry Kendall College, Tulsa, Okla., Dr. W. E. HOWARD was appointed in charge of the department of mathematics. Mr. O. S. DUFFENDICK, who had been in charge of the departments of mathematics and physics and who is during the present year on leave of absence, will return next year to take charge of the department of physics.

Mr. C. J. STOWELL was appointed in charge of the department of mathematics at McKendree College, Lebanon, Ill.

Mr. ALFRED DOOLITTLE, instructor of astronomy at the Catholic University of America, Washington, D. C. since 1913, died February 23, 1921. Son of the late Charles Leander Doolittle [*1919*, 178], and a brother of the late Eric Doolittle [*1920*, 438], he was born at Ontario, Ind., June 14, 1867. After receiving his A.B. from Lehigh University in 1887, he was instructor of mathematics and astronomy at Lehigh, 1889–91, assistant in the Nautical Almanac Office, 1892–97, instructor of mathematics and director of the astronomical observatory of The Catholic University of America, 1898–1901, 1906–1913. He was a piece-work computer at the Nautical Almanac Office from 1897 until his death.

Don TOMÁS DE AZCÁRATE, director of the Marine Observatory of San Fernando, Spain, since 1903, died on January 25, 1921, at the age of seventy-one years. He published at least one volume entitled *Anales del Instituto y Observatorio de Marina de San Fernando*, and had charge of the preparation of the *Almanaque Nautico*.

WILHELM JULIUS FÖRSTER, dean of German astronomers, died on January 18, 1921. He was born December 16, 1832. He joined the staff of the Berlin Observatory in 1855, became its director ten years later, and retired in 1904. "He

was one of the first two secretaries of the Astronomischer Gesellschaft, and served his country and the scientific world faithfully both there and in many other works. He was an active supporter of the Geodetisches Institut, and in connection both with that and with astronomical conferences he had a world wide circle of friends who will mourn his loss" (*The Observatory*, April, 1921).

MAGNUS NYRÉN, for the forty years 1868–1908 a worker at the Pulkowa Observatory, Russia, died January 16, 1921. Born in Sweden February 21, 1837, his earliest astronomical work was at the Upsala Observatory. He was the author of a large number of important papers (see "Poggendorff"). "After his retirement he lived in Stockholm, and last year published a catalogue of the proper motions of 633 stars in the new Pulkowa fundamental catalogue" (*Observatory*, March, 1921).

GUSTAV WILHELM LUDWIG STRUVE, younger brother of K. H. Struve whose death we have recently recorded (*1920*, 438) died November 4, 1920. Born October 20, 1858, he was astronomer in the Pulkowa Observatory 1880–1894. In 1894 he was made extraordinary professor and in 1897 ordinary professor of astronomy and geodesy at the University of Charcow. His best known work was on the constant of precession, but he took an active part in the Russian triangulation.

OSKAR LESSER, of the Klingeroberrealschule in Frankfurt a.M., Germany, who died September 23, 1920, was born October 4, 1867. He was the author of several books dealing with elementary mathematics through the calculus. For example: *Hilfsbuch für den geometrischen Unterricht an höheren Lehranstalten*, 1902; *Die Infinitesimalrechnung im Unterricht der Prima*, 1906 (second edition, 1911); and *Graphische Darstellungen im Mathematikunterricht der höheren Schulen*, 1908. There were also numerous editions and forms of K. Schwab and O. Lesser's *Mathematisches Unterrichtswerk*, of from one to three volumes each, 1909–1915. The editor of *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* has very kindly sent to us proof sheets of a five-page article on Oskar Lesser by G. Wolff. It concludes with a list of 19 titles of his publications.

Professor E. L. MARK, director of the Zoölogical Laboratory at Harvard University, has just retired from active teaching, and been appointed professor emeritus, after forty-four years of service on the staff of the University. In 1871 he received the degree of A.B. from the University of Michigan and taught mathematics there during the next year. He then served for a year as astronomer of the United States Northwest Boundary Survey. In 1873 he went to Europe. Returning in 1877 he became instructor in zoölogy at Harvard University.

At Cornell University Professor VIRGIL SNYDER has been granted leave of absence for the academic year 1921–1922. For the second term of the year he has been awarded, from the Heckscher Foundation for the promotion of research at Cornell University, a grant in support of research in the theory of algebraic surfaces. To Professor H. S. VANDIVER has been awarded a grant from the same Foundation in support of investigations on the theory of algebraic numbers. The Heckscher Foundation was established in 1920 by a gift to Cornell University of five hundred thousand dollars. The donor was August Heckscher, the New

York capitalist, miner, and manufacturer, born at Hamburg, Germany, in 1848. The plan for the administration of the Foundation was published in *Science*, October 15, 1920.

Of the five thousand dollars set aside for distribution in 1921 by the Committee on Grants of the American Association for the Advancement of Science, the following amounts were assigned for work in the fields of mathematics and astronomy: one hundred and fifty dollars to Professor SOLOMON LEFSCHETZ, of the University of Kansas, in support of his work in algebraic geometry; two hundred dollars to Dr. SEBASTIAN ALBRECHT, of Dudley Observatory, Albany, N. Y., in support of his investigation of the variation of wave-length of lines in different types of stellar spectra; and two hundred dollars to Miss CAROLINE E. FURNESS, of Vassar College Observatory, for assistance in the measurement and reduction of photographic plates.

Sir GEORGE GREENHILL was last March elected a corresponding member of the section in mechanics of the Academy of Sciences of the Institute of France (see *1920*, 384; *1921*, 238).

On April 20, 1921, Professor OSWALD VEBLEN was elected for the ensuing year a member of the executive committee of the Division of Physical Sciences of the National Research Council (cf. *1920*, 190).

In April, 1921, Professor G. A. MILLER, of the University of Illinois, was elected a member of the National Academy of Sciences in the Section of Mathematics. The other thirteen members of the Section are: G. D. BIRKHOFF, G. A. BLISS, H. F. BLICHFELDT, OSKAR BOLZA, L. E. DICKSON, EDWARD KASNER, E. H. MOORE, W. F. OSGOOD, W. E. STORY, E. B. VanVLECK, OSWALD VEBLEN, H. S. WHITE, and E. J. WILCZYNSKI. A. R. FORSYTH, DAVID HILBERT, M. E. C. JORDAN, C. F. KLEIN, JOSEPH LARMOR, C. E. PICARD, and VITO VOLTERRA are foreign associates of the Academy. Professor A. E. KENELLY, of Massachusetts Institute of Technology, was also elected a member, in the Section of Engineering.

In April, 1921, G. D. BIRKHOFF, of Harvard University, was elected a member of the American Philosophical Society. Among the members of this Society are the following mathematicians and astronomers: E. W. BROWN, W. W. CAMPBELL, L. E. DICKSON, L. P. EISENHART, H. B. FINE, G. E. HALE, E. O. LOVETT, J. A. MILLER, E. H. MOORE, FRANK MORLEY, F. R. MOULTON, W. F. OSGOOD, H. N. RUSSELL, FRANK SCHLESINGER, T. J. J. SEE, M. B. SNYDER, OSWALD VEBLEN, A. G. WEBSTER, E. B. WILSON, and R. S. WOODWARD. Among the foreign members are JOSEPH LARMOR, C. E. PICARD and VITO VOLTERRA.

Foreigners of the allied nations are now admitted as students in residence (élèves internes) at the Ecole Polytechnique, Paris.

The University of Nebraska offers to graduate students three fellowships in mathematics, of one thousand dollars each, for 1921–22. These fellowships are open to graduates of any good college or university. Appointees will be expected to devote about half their time to teaching.

At the annual meeting in Spokane the last of March the Mathematics Section of the Inland Empire Teachers' Association was reorganized as the Inland Empire Council of Teachers of Mathematics, with the intention of affiliating more closely with the National Council of Teachers of Mathematics. Professor W. C. EELLS of Whitman College, Walla Walla, Washington, was chosen first President of the

new organization. The greater part of the meeting was devoted to a discussion of the report, of the National Committee, on College Entrance Requirements in Mathematics. The sentiment of the teachers present was very favorable to the general features of this report.

At the meeting of the Academy of Sciences of the Institute of France held December 27, 1920, a paper by L. E. DICKSON was presented. It was entitled: "Les polynomes égaux à des déterminants." At the meeting of March 14, 1921, the following papers were presented: "La composition des polynomes" by L. E. DICKSON, and "Sur la position des racines des dérivées d'un polynome" by J. L. WALSH.

At recent meetings of the London Mathematical Society the following papers were communicated by title from the chair. On November 11, 1920: "On the conformal transformations of a space of four dimensions" by H. BATEMAN; "Arithmetic of quaternions" by L. E. DICKSON; and "The group of the linear continuum" by N. WIENER. On January 13, 1921: "Determination of all the characteristic sub-groups of an Abelian group" by G. A. MILLER.

At the spring meeting of The Association of Teachers of Mathematics in New England, held in Boston University, May 7, 1921, the following program was given: "Models for teaching geometry" by A. H. WHEELER, North High School, Worcester, Mass.; "Junior high school mathematics" by J. A. FOBERG, Pennsylvania State Department of Education; "Proportional representation" by E. V. HUNTINGTON, Harvard University; "Teaching arithmetic, algebra, and geometry in elementary and secondary schools, together and concretely" by C. W. ELIOT, president emeritus, Harvard University.

Among the papers presented at meetings of the American Philosophical Society, Philadelphia, April 21–23, were the following: "Further investigations on the relation between terrestrial magnetism and atmospheric electricity" by L. A. BAUER; "Discussion of a kinetic theory of gravitation" and "Some new experiments in gravitation" by C. F. BRUSH; "The atomic theory and ideal numbers" by L. E. DICKSON; "Discussion of the application of the method of the interferometer to the measurement of double stars" by J. A. MILLER; "Recent astronomical explorations in space and in time" by F. R. MOULTON; "The Roger Bacon cipher" by W. R. NEWBOLD.

At the annual meeting of the National Academy of Sciences held in Washington, April 25–26, 1921, the following papers were read: "On the problem of three or more bodies" by G. D. BIRKHOFF; "On the approximate solutions in integers of a set of linear equations" by H. F. BLICHFELDT; "Investigations in algebra and number theory" and "Quaternions and their generalizations" by L. E. DICKSON; "A model of the solar gravitational field" by EDWARD KASNER; "On the problem of steering an automobile around a corner," "On the radiation of energy from coils in wireless telegraphy," and "On the vibrations in gun-barrels" by A. G. WEBSTER. Dr. ALBERT EINSTEIN was present at public sessions of the Academy, and responded to an address of welcome presented to him by President Walcott.

At the April meeting of the San Francisco Section of the American Mathematical Society held at Stanford University on Saturday, April 9, the following papers were presented: "Anharmonic polynomial generalizations of the numbers

of Bernoulli and Euler," "Note on the prime divisors of the numerators of Bernoulli's numbers," "On a general arithmetical formula of Liouville," "Proof of an arithmetical theorem due to Liouville," and "The Bernoullian functions occurring in the arithmetical applications of elliptic functions" by E. T. BELL; "The approximate solutions in integers of a set of linear equations" by H. F. BLICHFELDT; "Euclid of Alexandria and the bust of Euclid of Megara," and "The spread of Newtonian and Leibnizian notations of the calculus" by F. CAJORI; "Normal ternary continued fractions" by P. H. DAVIS; "Autopolar curves and surfaces" by M. W. HASKELL; "On the computation of interest on certain kinds of investments" by D. N. LEHMER; "The hyperspace generalization of the lines on the cubic surface" by D. V. STEED.

At the sixteenth regular western meeting of the American Mathematical Society, held in Chicago, March 25 and 26, 1921, the following papers were presented: "New properties of all functions" by H. BLUMBERG; "Fallacies and misconceptions in Diophantine analysis" and "A new method in Diophantine analysis" by L. E. DICKSON; "Some new formulæ in combinatory analysis" by A. DRESDEN; "The group of motions of an Einstein space" by J. EIESLAND; "A disputed point regarding the nature of the continuum" by W. B. FORD; "Summable infinite determinants" by W. L. HART; "On a general theory of functions" (preliminary communication) by T. H. HILDEBRANDT; "The general theory of approximation by polynomials and trigonometric sums" by D. JACKSON; "A general theory of congruences" by E. P. LANE; "The equivalence of pairs of Hermitian forms" by MAYME I. LOGSDON; "Invariants and vector covariants of linear algebras without the associative law" by C. C. MACDUFFEE; "An overlooked infinite system of groups of order pq^2, p and q being prime numbers" by G. A. MILLER; "On the expansion of powers of trigonometric functions" and "On the summation of a trigonometric power-series" by I. J. SCHWATT; "Generational definition of linear associative hypernumbers" and "On Hamiltonian products" (second paper) by J. B. SHAW; "On non-loxodromic substitution groups in n dimensions" by E. B. VAN VLECK; "Some projective generalizations of geodesics" by E. J. WILCZYNSKI; "Congruences characterized by certain coincidences" by F. E. WOOD.

At the meeting of the American Mathematical Society in New York City, April 23, 1921, the following papers were presented: "A covariant of three circles" by A. B. COBLE; "Most general composition of polynomials" and "Number of real roots by Descartes' rule of signs" by L. E. DICKSON; "Concerning Laguerre's inversion" and "On certain type of system of ∞^2 curves" by J. DOUGLAS; "The Einstein solar field" by L. P. EISENHART; "The kernel of the Stieltjes integral corresponding to a completely continuous transformation" by C. A. FISCHER; "The mathematical theory of proportional representation" (third paper) by E. V. HUNTINGTON; "Topics in the theory of divergent series" by W. A. HURWITZ; "Note on an irregular expansion problem" by D. JACKSON; "Closed connected point sets which are disconnected by the omission of a finite number of points" by J. R. KLINE; "On the geometry of motion in a curved space of n-dimensions" by J. LIPKA; "Hyperspherical goniometry, with applications to the theory of correlation for n variables" by J. MACMAHON; "On the gyroscope" by W. F. OSGOOD; "On the apportionment of representatives" (second paper) by F. W. OWENS; "On the theorems of Green and Gauss" by

V. C. POOR; "On a simple class of deductive systems" by E. L. POST; "Pressure distribution around a breech-block" by J. E. ROWE; "Higher derivatives of functions of functions," "Method for the summation of a general case of a deranged series" and "The sum of a series as the solution of a differential equation" by I. J. SCHWATT; "On the location of the roots of polynomials" by J. L. WALSH; "Seven points in space and the eighth associated point" by H. S. WHITE; "A special kind of ruled surface" by J. K. WHITTEMORE; "A new vector method in integral equations" by N. WIENER and F. L. HITCHCOCK.

Professor ALBERT EINSTEIN arrived in the United States on April 2. Although he came primarily in the interests of the Zionist movement [*1921*, 191], he has been giving scientific lectures, in German, at various universities. On April 15 he lectured at Columbia University; on April 18, 19, 20, and 21 at the College of the City of New York; on May 3, 4, and 5 at the University of Chicago, and on May 9, 10, 11, 12, and 13 at Princeton University.

Courses in applied mathematics to be offered at the *Massachusetts Institute of Technology* during the year 1921–22 are as follows: Mathematical theory of investment, by Dr. J. S. TAYLOR, Fourier's series, by Dr. NORBERT WIENER, Vector analysis, by Dr. S. D. ZELDIN, Statistical mechanics, by Dr. H. B. PHILLIPS, Aeronautics, by Dr. C. L. E. MOORE, and Applied mathematics to chemistry, by Dr. F. L. HITCHCOCK (cf. *1920*, 242).

Additional courses in mathematics offered at Summer Sessions in 1921 (cf. *1921*, 194, 241) are as follows:

University of Colorado, First Term, June 13–July 20; Second Term, July 21–August 27. A review course in mathematics, by Professor G. H. LIGHT, will be given the first term only. The following courses will be given throughout the quarter: College algebra, by Professor LIGHT; Trigonometry, Plane analytic geometry, and Differential and integral calculus, by Professor G. W. SMITH, of the University of Kansas; Differential equations, by Professor ABRAHAM COHEN, of The Johns Hopkins University. The following term courses will be given either the first or second term: Teachers' course in mathematics, by Professor COHEN; Theory of equations, by Professor LIGHT. One or more of the following term courses will be given by Professors COHEN and LIGHT to meet the demands of the greatest number: Calculus of variations, Definite integrals, Theory of a complex variable, Elliptic integrals and functions, An introductory course in analysis, Differential geometry, and Series. Each course meets five days a week. Five hours credit is given for a course throughout the quarter; two and one-half hours credit for a term course.

University of Wisconsin, June 28–August 5. By Professor C. N. SLICHTER: Mechanics, 5 hours, and Applications of calculus, 3 hours. By Professor E. B. VANVLECK: Survey of algebra and geometry, 5 hours, Analytic geometry, 5 hours, and Theory of integrals including applications to Beta and Gamma functions, 3 hours. By Professor L. W. DOWLING: Plane trigonometry, 5 hours, Projective geometry, 5 hours, and Geometrical aspects of relativity, 3 hours. By Professor WARREN WEAVER: Theory of equations, 5 hours, Differential equations, 5 hours, and Wave theory, 3 hours. By Professor W. W. HART:

Algebra, 5 hours, and Teachers' course, 5 hours. By Mr. J. E. DAVIS: Differential calculus, 12 hours. By Mr. M. H. INGRAHAM: Integral calculus, 12 hours. By Mr. FREDERICK WOOD: Elementary analysis, 12 hours.

THE SUMMER MEETING OF THE ASSOCIATION.

The sixth summer meeting of the Association will be held at Wellesley College, Wellesley, Mass., on Tuesday and Wednesday, September 6–7, 1921. It will immediately precede the meeting of the American Mathematical Society to be held at the same place September 7–9. A joint meeting of the two organizations will be held on Wednesday afternoon, when phases of relativity will be the topic for discussion, and there will be a joint dinner Wednesday evening. Detailed programs of the meeting will be mailed to all members of the Association at a later date. Tower Court will be available for the accommodation of attending members and friends at a rate of three dollars per day for room and meals. A separate wing will be reserved for ladies and married couples. The great hall of Tower Court will furnish an attractive meeting place for those in attendance. Several social functions and perhaps an excursion will be arranged. Lake Waban will be available for boating and bathing. Wellesley is fifteen miles west of Boston on the main line of the Boston and Albany Railroad.

ANNALS OF MATHEMATICS

The following notice appearing in the current issue of the *Annals of Mathematics* is commended to the attention of members of the Association: "According to an agreement between the Mathematical Association of America and the editors of the *Annals of Mathematics*, the Association contributes to the support of the *Annals*, and the *Annals* is supplied to individual members of the Association at one half of the regular price. In consequence of this agreement the volume of the *Annals* was increased by 100 pages which are devoted to expository and historical articles in so far as suitable articles of this class are obtainable. Thus far the editors have not received enough such articles to fill the space available, and therefore wish to call the attention of authors to this lack and to the fact that as long as the shortage continues expository or historical articles of sufficient merit will receive prompt publication.

"A number of the expository articles which have already been published are available in separate form and are listed for sale on the inside of the back cover of this number of the *Annals*. The regular subscription price of the *Annals* is $3.00 a volume."

At present 241 members of the Association are subscribers to the *Annals*.

[*Copy sent to the printer, April 23, 1921; printing delayed by a strike till August 15, 1921.*]

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of ..... Dollars, to be known as the ..... Fund, and to be used[2] for { Endowment—the income only of which may be expended. / Special Projects—for which both principal and income may be expended.

Witness: Signature .....

.....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

CONTENTS

The March Meeting of the Rocky Mountain Section. By Professor G. H. LIGHT 243
Rational Triangles and Quadrilaterals. By Professor L. E. DICKSON . . . 244
The Triangle of Reference in Elementary Geometry. By Professor LENNIE P. COPELAND 250
Among My Autographs: 8. Francœur Describes a King; 9. LeVerrier and the Cost of Living. By Professor D. E. SMITH 254
The Formula $\frac{1}{2}a(a+1)$ for the Area of an Equilateral Triangle. By Professor G. A. MILLER 256
Note on the Prime Divisors of the Numerators of Bernoulli's Numbers. By Professor E. T. BELL 258
QUESTIONS AND DISCUSSIONS: Questions—**43, 44.** Discussions—"The intersection of two conic sections with a common focus" by Dr. H. F. MAC NEISH; "An arithmetical perpetual calendar" by Dr. PHILIP FRANKLIN . . . 260
RECENT PUBLICATIONS: Reviews by Professor ARCHIBALD, etc. Notes. Articles in Current Periodicals 263
UNDERGRADUATE MATHEMATICS CLUBS: Club activities—Connecticut College; Goucher College; Harvard University; University of Minnesota; Rockford College; University of Texas. Notes 272
PROBLEMS AND SOLUTIONS: Problems for solution—**2900-2907.** Problems–Notes—**16-18.** Problems–Solutions—**2801, 2805, 2806,** 2822 277
NOTES AND NEWS 285

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS Oberlin, Ohio.

Sixth Annual Meeting, Univ. of Toronto, December, 1921
Sixth Summer Meeting of the Association, Wellesley College, September 6-7, 1921
The following are dates of Section meetings of the Association in 1920, unless otherwise specified

ILLINOIS, Univ. of Chicago, December 29
IOWA, Iowa City, May 1; Des Moines, November 5; Simpson College, Indianola, April 30, 1921.
KANSAS, State Agricultural College, Manhattan, April 3; Topeka, January 22, 1921
KENTUCKY, Danville, April 17; University of Kentucky, May 7, 1921
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, May 15; Naval Academy, Annapolis, Md., December 11; Washington, D. C., May 7, 1921
MINNESOTA, St. Catherine's College, St. Paul, June 5; College of St. Thomas, St. Paul, June 4, 1921
MISSOURI, Kansas City Junior College, November 13; Soldan High School, St. Louis November 25–26, 1921
OHIO, Columbus, April 2; Columbus, March 25–26, 1921
ROCKY MOUNTAIN, Colorado College, Colorado Springs, April 2; Denver, March 25–26, 1921

Volumes and Sets of the Monthly

Complete sets of the Monthly (1894–1920) are obtainable only occasionally through dealers in periodicals, but many single numbers and complete volumes (1894–1912) may be had through the Secretary at varying prices, according to scarcity of stock.

Volumes for 1913, 1914 and 1915 will be sold, when available, *only to members of the Association who can thereby make up complete sets*—price, $5.50.

Most of the volumes for 1916–1920 can be obtained through the Secretary at $5.00, but scarcity of a few issues here also will raise the price of certain volumes to $6.00.

Address all communications to the Secretary,

W. D. CAIRNS
Oberlin, Ohio